Derivatives الاشتقاقات

Defn. Let $y=f(x)$ be a function of x. If the limit

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$ exist and define, we call it the derivative of f at x (or f is differentiable at x) and denoted by $\frac{dy}{dx}$, $\frac{df}{dx}$. That is

$$f'(x)=y'=\frac{dy}{dx}=\frac{df}{dx}=\lim_{\Delta x \to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}, \quad \Delta x \neq 0$$

Rules of Derivatives : قواعد الاشتقاق

Rule(1) If $y=f(x)=C$, where C is constant, then

$$\frac{dy}{dx}=f'(x)=0 .$$

Rule(2) If is +ive integer and $y=f(x)=x^n$, then

$$f'(x)=\frac{dy}{dx}=n\,x^{n-1}$$

Rule(3) If $f(x)=c\,u(x)$, where c is constant and $u(x)$ is a differentiable of x then $f'(x)=c\,u'(x)$ or $\frac{dy}{dx}=c\frac{du}{dx}$.

Rule(4) If $u_i(x)$, $i=1,2,3,\dots,n$ are differentiable functions of x and $f(x)=u_1(x)\pm u_2(x)\pm\dots\pm u_n(x)$, then

$$f'(x)=u_1'(x)\pm u_2'(x)\pm\dots\pm u_n'(x).$$

Rule(5) If $y=f(x)=u(x)\cdot v(x)$, where $u(x)$ and $v(x)$ are differentiable functions of x, then $f'(x)=u(x)\,v'(x)+v(x)\,u'(x)$.

or $$\frac{dy}{dx}=u(x)\cdot\frac{dv}{dx}+v(x)\cdot\frac{du}{dx}.$$

Rule(6) If $f(x) = \frac{u(x)}{v(x)}$, $v(x) \neq 0$ where $u(x)$ and $v(x)$ are differentiable functions of x then

$$f'(x) = \frac{v(x)\,u'(x) - u(x)\,v'(x)}{[v(x)]^2} \quad \text{or} \quad \frac{dy}{dx} = \frac{v(x)\frac{du}{dx} - u(x)\frac{dv}{dx}}{[v(x)]^2}.$$

Rule(7) If $f(x) = [u(x)]^n$ where n is +ive integer and $u(x)$ is a differentiable function of x, then

$$f'(x) = n\,[u(x)]^{n-1}\,u'(x) \quad \text{or} \quad \frac{dy}{dx} = n\,[u(x)]^{n-1}\frac{du}{dx}.$$

Rule(8) If $f(x) = [u(x)]^n$ where n is −ive integer and $u(x)$ is a differentiable function of x then

$$f'(x) = n\,[u(x)]^{n-1}\,u'(x).$$

Rule(9) If $f(x) = [u(x)]^{\frac{p}{q}}$, where $u(x)$ is a differentiable function of x with p and q are integers $(q \neq 0)$, then

$$f'(x) = \frac{p}{q}[u(x)]^{\frac{p}{q}-1}\,u'(x).$$

Implicit Differentiations :

Consider the function defined by the eq. $f(x,y) = 0$ which may or may not be solved for y in terms of x.

for example $y - x^3 + 2x - 5 = 0$ Can be written as

$$y = x^3 - 2x + 5 \quad \text{and} \quad \frac{dy}{dx} = 3x^2 - 2.$$

While $y^5 + 4x^2y^2 + x^3 - 2 = 0$ Can not be solved for y in terms of x.

Implicit differentiation enables us to find the derivative of such functions whenever they exist.

EX. Find $\frac{dy}{dx}$ if $y^3 - 3x^2y + x^3 = 5$

$$3y^2\frac{dy}{dx} - \left[3x^2\frac{dy}{dx} + y(6x)\right] + 3x^2 = 0$$

$$3y^2\frac{dy}{dx} - 3x^2\frac{dy}{dx} - 6xy + 3x^2 = 0$$

$$(3y^2 - 3x^2)\frac{dy}{dx} = 6xy - 3x^2 \Rightarrow \frac{dy}{dx} = \frac{3(2xy - x^2)}{3(y^2 - x^2} = \frac{2xy - x^2}{y^2 - x^2}$$

The second and Higher Derivatives :

Given the function $y = f(x)$. The derivative $y' = f'(x) = \frac{dy}{dx} = \frac{df}{dx}$ is the 1st derivative of y w.r.to x

and $y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{d^2f}{dx^2}$ is called the 2nd derivative w.r.to x

Thus the second derivative is the derivative of the first derivative. That is

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

In general, if $y = f(x)$ is a differentiable function of x, then the nth derivative of y w.r.t. x is ~~defined~~ denoted by:

$$f^{(n)}(x) = y^{(n)} = \frac{d^ny}{dx^n}.$$

Ex. If $y=(3x^3+2x-1)^{1/2}$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

Solu.

$$\frac{dy}{dx}=\frac{1}{2}(3x^3+2x-1)^{-\frac{1}{2}}(9x^2+2)$$

$$\frac{d^2y}{dx^2}=\frac{1}{2}\left[(3x^3+2x-1)^{-\frac{1}{2}}(18x)+(9x^2+2)\left[-\frac{1}{2}(3x^3+2x-1)^{-3/2}(9x^2+2)\right.\right.$$

Ex. If $y=3x^4-5x^3+6x-7$. Find $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots, \frac{d^ny}{dx^n}$

Solu.

$$\frac{dy}{dx}=12x^3-15x^2+6$$

$$\frac{d^2y}{dx^2}=36x^2-30x \Rightarrow \frac{d^3y}{dx^3}=72x-30 \Rightarrow \frac{d^4y}{dx^4}=72$$

$$\frac{d^5y}{dx^5}=\frac{d^6y}{dx^6}=\dots=\frac{d^ny}{dx^n}=0.$$

H.W
Ex. If $x^2-y^2=1$. Show that $\frac{dy}{dx}=\frac{x}{y}$ and $\frac{d^2y}{dx^2}=-\frac{1}{y^3}$.

Ex. If $y=(2x^2-5x^{-2})^{-5}$. Find $\frac{dy}{dx}$

$$\frac{dy}{dx}=-5(2x^2-5x^{-2})^{-6}(4x+10x^{-3}).$$

H.W
Ex. Find $\frac{d}{dx}\left[\frac{(x^2+x+1)^3}{(x^3+1)^4}\right]$

H.W
Ex. Find $\frac{dy}{dx}$, if $y=\frac{2x^3+3x-1}{x^2+1}$.

H.W
Ex. Find $\frac{dy}{dx}$, if $y=(x^2+2)(x^3+3x+1)$.

Chain Rule And Parametric Equations

<u>Chain Rule</u> If y is a function of x, say $y=f(x)$, and x is a function of t, say $x=g(t)$, then y is a function of t and

$$\boxed{\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}} \quad \cdots (1)$$

How we obtain eq. (1)

Since $y=f(x)$ and $x=g(t)$, then

$$y=f[g(t)] \text{ and } \frac{dy}{dt}=f'[g(t)]\cdot g'(t) ,$$

but $x=g(t)$ and $\frac{dx}{dt}=g'(t)$

$$\therefore \frac{dy}{dt}=f'(x)\cdot\frac{dx}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt} .$$

<u>EX.</u> If $y=x^3-2x^2+3$ and $x=t^2+2$. Find $\frac{dy}{dt}$ at $t=2$.

<u>Solu.</u>

$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}=(3x^2-4x)(2t)$$

when $t=2 \Rightarrow x=(2)^2+2=6$

$$\therefore \left.\frac{dy}{dt}\right|_{t=2}=[3(36)-4(6)=(108-24)(4)=336 .$$

<u>or</u> $y=(t^2+2)^3-2(t^2+2)^2+3 \Rightarrow \frac{dy}{dt}=3(t^2+2)(2t)-4(t^2+2)(2t)$

$$\therefore \left.\frac{dy}{dt}\right|_{t=2}=3(36)(4)-4(6)(4)=336 .$$

Parametric Equations

Sometimes, we may describe the curve by expressing both coordinates as functions of a third variable, say $x=g(t)$ and $y=f(t)$.

These two eq.'s are called the parametric eq.'s for x and y and the variable t is called a parameter.

Ex. Determine an equation in x and y of the following parametric eq.'s and then find $\frac{dy}{dx}$.

(a) $y=\frac{1}{t}$, $x=t \Rightarrow y=\frac{1}{x} \Rightarrow \frac{dy}{dx}=-\frac{1}{x^2}$.

(b) $y=t^2$, $x=\frac{t}{1-t}$

$$x=\frac{t}{1-t} \Rightarrow x-xt=t \Rightarrow x=t+xt \Rightarrow x=t(1+x)$$

$$\Rightarrow t=\frac{x}{1+x} \quad \therefore y=\left(\frac{x}{x+1}\right)^2 \Rightarrow \frac{dy}{dx}=2\left(\frac{x}{x+1}\right)\left(\frac{x+1-x}{(x+1)^2}\right)$$

$$\therefore \frac{dy}{dx}=\frac{2x}{(x+1)^3}.$$

(c) $y=2\sin t$, $x=3\cos t$

$$\sin t=\frac{y}{2} \quad , \cos t=\frac{x}{3} \Rightarrow \sin^2 t+\cos^2 t=\frac{y^2}{4}+\frac{x^2}{9}=1$$

$$\Rightarrow \frac{2y}{4}\frac{dy}{dx}+\frac{2x}{9}=0 \Rightarrow \frac{y}{2}\frac{dy}{dx}=-\frac{2}{9}x \Rightarrow \frac{dy}{dx}=-\frac{4}{9}\frac{x}{y}.$$

(d) $y=\frac{5(4-t^2)}{4+t^2}$, $x=\frac{20t}{4+t^2}$

$$y^2+x^2=\frac{25(4-t^2)^2}{(4+t^2)^2}+\frac{400t^2}{(4+t^2)^2}=\frac{25(16-8t^2+t^4)+400t^2}{(4+t^2)^2}$$

$$= \frac{25(16-8t^2+t^4+16t^2)}{(4+t^2)^2} = \frac{25(16+8t^2+t^4)}{(4+t^2)^2}$$

$$= \frac{25(4+t^2)}{(4+t^2)^2}$$

$$\therefore\ y^2+x^2=25 \Rightarrow 2yy'+2x=0 \Rightarrow y'=\frac{-x}{y}.$$

H.W (e) $y=t^2$, $x=t-1$, H.W (f) $y=2+2\sin t$, $x=-1+2\cos t$.

H.W (g) $y=3\tan t$, $x=4\sec t$ H.W (h) $y=\sin^3 t$, $x=\cos^3 t$

H.W (i) $y=t^2+t-1$, $x=t^2+2t+3$ H.W (j) $y=\frac{3(2-t)(2+t)^2}{16t^2+8}$, $x=\frac{3(2-t)^2(2}{6t^2+8}$

Derivatives of the Parametric Eq.'s

The 1st derivative

If $y=f(t)$ and $x=g(t)$, then

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{dy}{dt}\cdot\frac{dt}{dx} \Rightarrow \boxed{\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}}$$

EX. If $y=t^2-1$ and $x=2t+3$. Find $\frac{dy}{dt}$, $\frac{dx}{dt}$ and $\frac{dy}{dx}$

Solu. $\frac{dy}{dt}=2t$, $\frac{dx}{dt}=2$, $\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=(2t)\cdot\frac{1}{2}=t$

But $x=2t+3 \Rightarrow t=\frac{x-3}{2}$, $\therefore \frac{dy}{dx}=\frac{x-3}{2}$.

Ex. $y = t - \frac{1}{t}$, $x = t + \frac{1}{t}$

$$\frac{dy}{dt} = 1 + \frac{1}{t^2} \quad , \quad \frac{dx}{dt} = 1 - \frac{1}{t^2} \quad , \quad \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}}$$

$$\therefore \frac{dy}{dx} = \frac{t^2+1}{t^2-1} \Rightarrow \frac{dy}{dx} = \frac{t + \frac{1}{t}}{t - \frac{1}{t}} = \frac{x}{y} .$$

or $$y^2 - x^2 = \left(t - \frac{1}{t}\right)^2 - \left(t + \frac{1}{t}\right)^2 = \cancel{t^2} - 2 + \cancel{\frac{1}{t^2}} - \cancel{t^2} - 2 - \cancel{\frac{1}{t^2}}$$
$$= -4$$

$$\therefore y^2 - x^2 = -4 \Rightarrow 2y\frac{dy}{dx} - 2x = 0 \Rightarrow 2y\frac{dy}{dx} = 2x$$

$$\therefore \frac{dy}{dx} = \frac{x}{y} .$$

The 2nd derivative :

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{\frac{dx}{dt}} \cdot \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$$

$$= \frac{1}{\frac{dx}{dt}} \frac{\left(\frac{dx}{dt}\right)\cdot\left(\frac{d^2y}{dt^2}\right) - \left(\frac{dy}{dt}\right)\left(\frac{d^2x}{dt^2}\right)}{\left(\frac{dx}{dt}\right)^2}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{\left(\frac{dx}{dt}\right)\left(\frac{d^2y}{dt^2}\right) - \left(\frac{dy}{dt}\right)\left(\frac{d^2x}{dt^2}\right)}{\left(\frac{dx}{dt}\right)^3}}$$

Ex. If $x = t - t^2$, $y = t - t^3$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $t = 2$

Solu.

$$x = t - t^2 \Rightarrow \frac{dx}{dt} = 1 - 2t \Rightarrow \frac{d^2x}{dt^2} = -2 .$$

$$y = t - t^3 \Rightarrow \frac{dy}{dt} = 1 - 3t^2 \Rightarrow \frac{d^2y}{dt^2} = -6t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1-3t^2}{1-2t} \Rightarrow \left.\frac{dy}{dx}\right|_{t=2} = \frac{1-12}{1-4} = \frac{11}{3} .$$

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{dx}{dt}\right)\left(\frac{d^2y}{dt^2}\right) - \left(\frac{dy}{dt}\right)\left(\frac{d^2x}{dt^2}\right)}{\left(\frac{dx}{dt}\right)^3}$$

$$= \frac{(1-2t)(-6t) - (1-3t^2)(-2)}{(1-2t)^3} = \frac{-6t+12t^2+2-6t^2}{(1-2t)^3}$$

$$= \frac{6t^2-6t+2}{(1-2t)^3} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=2} = \frac{6(4)-6(2)+2}{(1-4)^3} = \frac{24-12+2}{(-3)^3}$$

$$= \frac{14}{-27}$$

or $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1-3t^2}{1-2t}\right) = \frac{\frac{d}{dt}}{\frac{dx}{dt}}\left(\frac{1-3t^2}{1-2t}\right)$$

$$= \frac{\frac{(1-2t)(-6t)-(1-3t^2)(-2)}{(1-2t)^2}}{(1-2t)}$$

$$= \frac{6t^2-6t+2}{(1-2t)^3} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=-2} = \frac{14}{-27}.$$

<u>Note</u> The chain Rule can be extended to many variables

In general: If $y=f_1(t_1)$, $t_1=f_2(t_2)$, $t_2=f_3(t_3)$, ..., $t_{n-1}=f_n(t_n)$, $t_n=f(x)$, then

$$\frac{dy}{dx} = \frac{dy}{dt_1}\cdot\frac{dt_1}{dt_2}\cdot\frac{dt_2}{dt_3}\cdot\ \cdots\ \cdot\frac{dt_{n-1}}{dt_n}\cdot\frac{dt_n}{dx}.$$

<u>EX.</u> If $y = x^3+2x^2+3x-4$. Find $\frac{dy^2}{dx^2}$.

<u>Solu.</u> Let $u=y^2$ and $v=x^2$

$$\frac{dy^2}{dx^2} = \frac{du}{dv} = \frac{du}{dy}\cdot\frac{dy}{dx}\cdot\frac{dx}{dv} = (2y)(3x^2+4x+3)\cdot\frac{1}{2x}$$

Ex. If $y=\frac{x}{x^2+1}$. Find $\frac{d\sqrt{y}}{d\sqrt{x}}$

Solu.

Let $u=\sqrt{y}$ and $v=\sqrt{x}$

$$\frac{d\sqrt{y}}{d\sqrt{x}}=\frac{du}{dv}=\frac{du}{dy}\cdot\frac{dy}{dx}\cdot\frac{dx}{dv}=\frac{1}{2\sqrt{y}}\cdot\left(\frac{(x^2+1)(1)-x(2x)}{(x^2+1)^2}\right)\cdot 2\sqrt{x}$$

Ex. Find $\frac{d\sqrt{x^2+1}}{d\,x^3}$.

Solu.

Let $u=\sqrt{x^2+1}$ and $v=x^3$

$$\frac{du}{dv}=\frac{du}{dx}\cdot\frac{dx}{dv}=\frac{2x}{2\sqrt{x^2+1}}\cdot\frac{1}{3x^2}$$

Indeterminate forms :

The meaningless $\frac{0}{0}$, $\frac{\infty}{\infty}$, 0^0, 1^∞, ∞^0, $0\cdot\infty$, $\infty-\infty$ are known as indeterminate forms.

Sometimes the limit $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ produce $\frac{0}{0}$ or $\frac{\infty}{\infty}$ when substituting $x=a$.

For example: $\lim\limits_{x\to 2}\frac{x^2-4}{x-2}=\frac{4-4}{2-2}=\frac{0}{0}$ meaningless.

So the solution is

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)}=\lim_{x\to 2}(x+2)=2+2=4.$$

L' Hopital's Rule (First Form) قاعدة اوبيتال

Suppose $f(a)=g(a)=0$ or ∞ and $f'(a)$ and $g'(a)$ exist with $g'(a)\neq 0$. Then $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}=\frac{f'(a)}{g'(a)}$.

Ex. (a) $\lim\limits_{x\to 2}\frac{x^2-4}{x-2}=\lim\limits_{x\to 2}\frac{2x}{1}=2(2)=4.$

(b) $\lim\limits_{x\to 0}\frac{x-2x^2}{3x^2+5x}=\lim\limits_{x\to 0}\frac{1-4x}{6x+5}=\frac{1-0}{0+5}=\frac{1}{5}$

(c) $\lim\limits_{x\to\infty}\frac{6x+5}{3x-8}=\lim\limits_{x\to\infty}\frac{6}{3}=\frac{6}{3}=2.$

(d) $\lim\limits_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\lim\limits_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}-0}{1}=\lim\limits_{x\to 0}\frac{1}{2\sqrt{1+x}}=\frac{1}{2}$

L' Hopital's Rule (Stronger Form):

Suppose $f(a)=g(a)=0$ or ∞ and the functions f and g with their derivatives are continuous in some interval I.

To find $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$, we proceed to differentiate f and g as long as we still get the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. But we stop the differentation as soon as one or the other derivatives is different from 0 or ∞ at $x=a$.

Note

L' Hopital's Rule does not apply when either numerator or denominator has a finite non-zero limit.

EX. (a) $\lim\limits_{x \to 1} \dfrac{x^3-3x+2}{x^3-x^2-x+1} = \lim\limits_{x \to 1} \dfrac{3x^2-3}{3x^2-2x-1}$

$= \lim\limits_{x \to 1} \dfrac{6x}{6x-2} = \dfrac{6(1)}{6(1)-2} = \dfrac{6}{4} = \dfrac{3}{2}.$

(b) $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x} - \frac{1}{2}x - 1}{x^2} = \lim\limits_{x \to 0} \dfrac{\frac{1}{2}(1+x)^{-1/2} - \frac{1}{2}}{2x}$

$= \lim\limits_{x \to 0} \dfrac{-\frac{1}{4}(1+x)^{-3/2}}{2} = -\dfrac{1}{8}.$

(c) $\lim\limits_{x \to \infty} \dfrac{2x^3-x^2+3x+1}{x^3+2x^2-x-1} = \lim\limits_{x \to \infty} \dfrac{6x^2-2x+3}{3x^2+4x-1}$

$= \lim\limits_{x \to \infty} \dfrac{12x-2}{6x+4} = \lim\limits_{x \to \infty} \dfrac{12}{6} = \dfrac{12}{6} = 2.$