"Transcendental Functions"

Trigonometric Functions الدوال المثلثية

Theorem (1) If $y = f(x) = \sin x$ then $\frac{dy}{dx} = f'(x) = \cos x$.

Proof

Let $h = \Delta x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{-\sin x\,(1 - \cos h) + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \left[\frac{-\sin x (1 - \cos h)}{h} + \frac{\cos x \sin h}{h} \right]$$

$$= -\sin x \lim_{h \to 0} \frac{1 - \cos h}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h}$$

$$= -\sin x \cdot (0) + \cos x \cdot (1) = \cos x.$$

Theorem (2) If $y = f(x) = \cos x$, then $\frac{dy}{dx} = f'(x) = -\sin x$.

Proof Let $h = \Delta x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h \to 0} \left[\frac{-\cos x (1 - \cos h}{h} - \frac{\sin x \sin h}{h} \right]$$

$$= \lim_{h\to 0} \frac{-\cos x(1-\cos h)}{h} - \lim_{h\to 0} \frac{\sin x \sin h}{h}$$

$$= -\cos x \lim_{h\to 0} \frac{1-\cos h}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h}$$

$$= -\cos x\,(0) - \sin(x)\,(1) = -\sin x .$$

Theorem (3)

(1) If $y = \tan x$ then $\frac{dy}{dx} = \sec^2 x$.

H.W (2) If $y = \cot x$ then $\frac{dy}{dx} = -\csc^2 x$.

H.W (3) If $y = \sec x$ then $\frac{dy}{dx} = \sec x \tan x$.

H.W (4) If $y = \csc x$ then $\frac{dy}{dx} = -\csc x \cot x$.

Proof (1)

$$y = \tan x = \frac{\sin x}{\cos x} \Rightarrow \frac{dy}{dx} = \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$\therefore \frac{dy}{dx} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x .$$

دالة قابلة للاشتقاق

Now, if $u = u(x)$ is a differentiable function of x and

1. $y = \sin u$ then $\frac{dy}{dx} = \cos u \cdot \frac{du}{dx}$.
2. $y = \cos u$ then $\frac{dy}{dx} = -\sin u \cdot \frac{du}{dx}$.
3. $y = \tan u$ then $\frac{dy}{dx} = \sec^2 u \cdot \frac{du}{dx}$.
4. $y = \cot u$ then $\frac{dy}{dx} = -\csc^2 u \cdot \frac{du}{dx}$.
5. $y = \sec u$ then $\frac{dy}{dx} = \sec u \tan u \frac{du}{dx}$.
6. $y = \csc u$ then $\frac{dy}{dx} = -\csc u \cot u \frac{du}{dx}$.

Ex(1) Find $\frac{dy}{dx}$ of the following :

(1) $y = \sin(x^2+2x-5) \Rightarrow \frac{dy}{dx} = \cos(x^2+2x-5) \cdot (2x+2)$

$= 2(x+1)\cos(x^2+2x-5)$.

(2) $y = \sin^2(x^2+\frac{1}{x^2}) \Rightarrow \frac{dy}{dx} = 2\sin(x^2+\frac{1}{x^2}) \cdot \cos(x^2+\frac{1}{x^2})(2x-\frac{2}{x^3})$

(3) $y = \tan(2x) \cdot \cos(x^2+1) \Rightarrow$

$\frac{dy}{dx} = -\tan(2x)\sin(x^2+1)(2x) + \cos(x^2+1)\sec^2(2x) \cdot 2$.

(4) $y = \tan^{-3}(3x^2+\sec^2 2x)$

$\frac{dy}{dx} = -3\tan^{-4}(3x^2+\sec^2 2x) \cdot \sec^2(3x^2+\sec^2 2x) \cdot (6x+2\sec 2x$

$\sec 2x \tan 2x \cdot 2)$.

(5) $y = \frac{\sec[\sin(2x+1)]}{\tan(x^3+1)}$

$\frac{dy}{dx} = \frac{\tan(x^3+1)\sec[\sin(2x+1)] \cdot \tan[\sin(2x+1)] \cdot \cos(2x+1)\,2 -}{\tan^2(x^3+1)}$

$\frac{\sec[\sin(2x+1) \cdot \sec^2(x^3+1) \cdot 3x^2}{\tan^2(x^3+1)}$.

Ex.(2) Find $\frac{d}{dx}[\sec^{-2}(x^2+2x) - \tan^2(\sin 3x)]$

$= -2\sec^{-3}(x^2+2x)\sec(x^2+2x)\tan(x^2+2x)(2x+2) - 2\tan(\sin 3x)$

$\sec^2(\sin 3x) \cdot \cos(3x) \cdot 3$.

EX.(3) Find $\frac{dy}{dx}$ if $x^2+5x-\tan^2(xy)=10$

Solu.

$$2x+5-2\tan(xy)\sec^2(xy)\cdot\left(x\frac{dy}{dx}+y\cdot(1)\right)=0$$

$$\therefore \frac{dy}{dx}=\frac{2x+5-2y\tan(xy)\sec^2(xy)}{2x\tan(xy)\sec^2(xy)}$$

EX.(4) Find the eq. of tangent to the Curve

$x\sin 2y = y\cos 2x$ at point $(\frac{\pi}{4},\frac{\pi}{2})$

Solu.

$$2x\cos 2y\cdot\frac{dy}{dx}+\sin 2y=-2y\sin 2x+\cos 2x\cdot\frac{dy}{dx}$$

$$2x\cos 2y\frac{dy}{dx}-\cos 2x\frac{dy}{dx}=-2y\sin 2x-\sin 2y$$

$$(2x\cos 2y-\cos 2x)\frac{dy}{dx}=-2y\sin 2x-\sin 2y$$

$$\therefore \frac{dy}{dx}=\frac{2y\sin 2x+\sin 2y}{\cos 2x-2x\cos 2y}$$ slope of tangent at any $P(x,y)$

$m=\frac{dy}{dx}$ at $x=\frac{\pi}{4}$, $y=\frac{\pi}{2}$

is $m=\dfrac{2(\frac{\pi}{2})\sin\frac{\pi}{2}+\sin\pi}{\cos\frac{\pi}{2}-2(\frac{\pi}{4})\cos\pi}=\dfrac{\pi(1)+0}{0-\frac{\pi}{2}(-1)}$

$\therefore m=2 \Rightarrow m=\dfrac{y-y_1}{x-x_1} \Rightarrow 2=\dfrac{y-\frac{\pi}{2}}{x-\frac{\pi}{4}}$ is the required eq.

EX.(5) If $f'(x)=\sin x^2$ and $y=f(\frac{2x+1}{x+1})$. Find $\frac{dy}{dx}$

Solu.

$$y=f\left(\frac{2x+1}{x+1}\right)\Rightarrow\frac{dy}{dx}=f'\left(\frac{2x+1}{x+1}\right)\cdot\frac{(x+1)(2)-(2x+1)(1)}{(x+1)^2}$$

$$\frac{dy}{dx}=\frac{1}{(x+1)^2}f'\left(\frac{2x+1}{x+1}\right)=\frac{1}{(x+1)^2}\sin\left(\frac{2x+1}{x+1}\right)^2.$$

EX. (6) If $y = \tan^{-3}(\sin 2x)$. Find $\dfrac{dy^2}{dx^2}$

Solu. Let $u = y^2$ and $v = x^2$

$$\frac{dy^2}{dx^2} = \frac{du}{dv} = \frac{du}{dy} \cdot \frac{dy}{dx} \cdot \frac{dx}{dv}$$

$$= 2y \cdot \left[-3 \tan^{-4}(\sin 2x) \sec^2(\sin 2x) \cos(2x) \cdot 2\right] \cdot \frac{1}{2x}$$

EX. (7) If $y = \sec 2t$ and $x = \csc 2t$. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at $t = \dfrac{\pi}{6}$

Solu.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2 \sec 2t \tan 2t}{-2 \csc 2t \cot 2t} = -\tan^3 2t$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{6}} = -\tan^3\left(\frac{\pi}{3}\right) = -(\sqrt{3})^3 = -3\sqrt{3}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}}{\frac{dx}{dt}}\left(-\tan^3 2t\right) = \frac{-3 \tan^2 2t \cdot \sec^2 2t \cdot 2}{-2 \csc 2t \cot 2t}$$

$$= -3 \tan^4 2t \sec 2t$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{6}} = -3 \tan^4\left(\frac{\pi}{3}\right) \sec\left(\frac{\pi}{3}\right) = -3 (\sqrt{3})^4 \cdot 2 = -54.$$

EX. (8) Evaluate the following Limits:

(1) $$\lim_{x \to 0} \frac{\sin 3x}{\sin 5x} = \lim_{x \to 0} \frac{3 \cos 3x}{5 \cos 5x} = \frac{3 \cos 0}{5 \cos 0} = \frac{3 \times 1}{5 \times 1} = \frac{3}{5}.$$

(2) $$\lim_{x \to 0} \frac{\sin 2x}{2x^2 + x} = \lim_{x \to 0} \frac{2 \cos 2x}{4x + 1} = \frac{2 \cos 0}{0 + 1} = \frac{2 \times 1}{1} = 2.$$

③ $$\lim_{x\to 0}\frac{\tan 3x}{\sin x}=\lim_{x\to 0}\frac{3\sec^2 3x}{\cos x}=\lim_{x\to 0}\frac{3\sec^2 0}{\cos 0}$$

$$=\frac{3\times(1)^2}{1}=3.$$

④ $$\lim_{x\to 0}\frac{x-\sin x}{x^3}=\lim_{x\to 0}\frac{1-\cos x}{3x^2}=\lim_{x\to 0}\frac{\sin x}{6x}$$

قاعدة اوستال ثلاث مرات ③

$$=\lim_{x\to 0}\frac{\cos x}{6}=\frac{\cos(0)}{6}=\frac{1}{6}$$

⑤ $$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\infty-\infty \quad \text{meaningless.}$$

قاعدة اوستال مرتين

$$=\lim_{x\to 0}\frac{x-\sin x}{x\sin x}=\lim_{x\to 0}\frac{1-\cos x}{x\cos x+\sin x\,(1)}$$

$$=\frac{1-\cos(0)}{0+\sin(0)}=\frac{1-1}{0+0}=\frac{0}{0}\ \text{meaningless}$$

$$=\lim_{x\to 0}\frac{\sin x}{x(-\sin x)+\cos x\,(1)+\cos x}=\frac{\sin(0)}{0+\cos(0)+\cos(0)}=\frac{0}{0+1+1}$$

$$=\frac{0}{2}=0.$$

⑥ $$\lim_{x\to 0}\frac{\sin x^2}{x\sin x}=\frac{0}{0}\ \text{meaningless}$$

$$=\lim_{x\to 0}\frac{\cos x^2\,(2x)}{x\cos x+\sin x\cdot(1)}=\frac{0}{0}\ \text{meaningless}$$

$$=\lim_{x\to 0}\frac{\cos x^2\,(2)+2x\,(-\sin x^2\,(2x))}{-x\sin x+\cos x+\cos x}=\frac{2+0}{0+1+1}=\frac{2}{2}=1.$$

سؤال امتحاني

⑦ $$\lim_{x\to\frac{\pi}{2}}(\sec x-\tan x)=\infty-\infty$$

$$=\lim_{x\to\frac{\pi}{2}}\left(\frac{1}{\cos x}-\frac{\sin x}{\cos x}\right)=\lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{\cos x}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{-\cos x}{-\sin x}=\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{\sin x}=\frac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}}=\frac{0}{1}=0.$$

The Inverse of Trigonometric Functions

معكوس الدوال المثلثية

Defn.

(1) For $-1 \leq x \leq 1$, we define the no. $y = f(x) = \sin^{-1} x$ for which $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and $x = \sin y$.

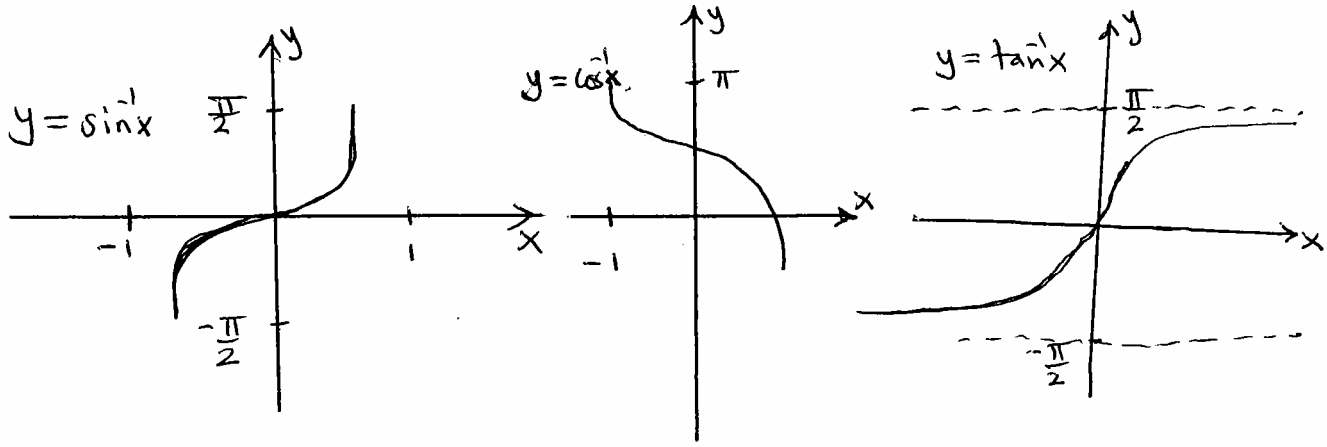
(2) For $-1 \leq x \leq 1$, we define the no. $y = f(x) = \cos^{-1} x$ for which $0 \leq y \leq \pi$ and $x = \cos y$.

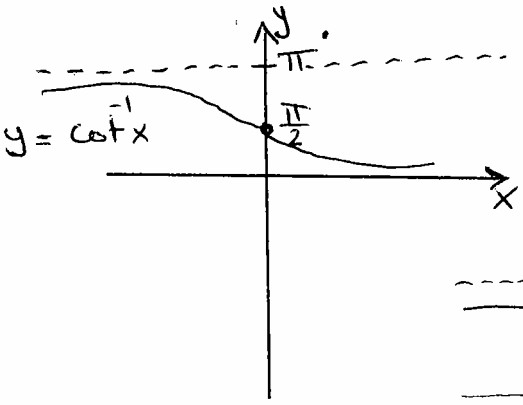
(3) For $-\infty < x < \infty$, we define the no. $y = f(x) = \tan^{-1} x$ for which $-\frac{\pi}{2} < y < \frac{\pi}{2}$ and $x = \tan y$.

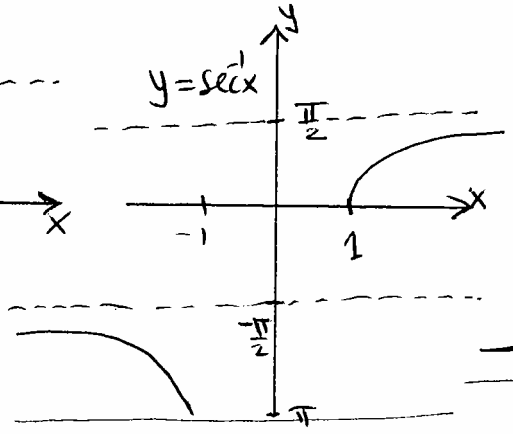
(4) For $-\infty < x < \infty$, we define the no. $y = f(x) = \cot^{-1} x$ for which $0 < y < \pi$ and $x = \cot y$.

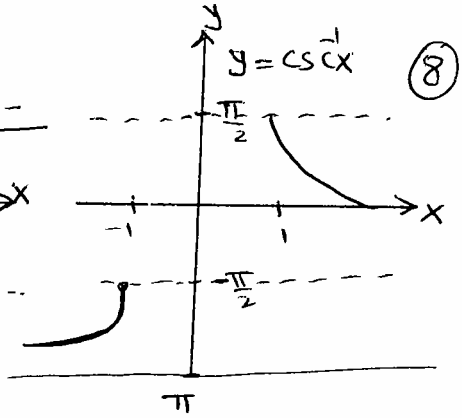
(5) For $x \leq -1$ or $x \geq 1$, we define the no. $y = f(x) = \sec^{-1} x$ for which $-\pi \leq y < -\frac{\pi}{2}$ or $0 \leq y < \frac{\pi}{2}$ and $x = \sec y$.

(6) For $x \leq -1$ or $x \geq 1$, we define the no. $y = f(x) = \csc^{-1} x$ for which $-\pi < y \leq -\frac{\pi}{2}$ or $0 < y \leq \frac{\pi}{2}$ and $x = \csc y$.









Note $\sin^{-1}x \neq \dfrac{1}{\sin x}$, $(\sin x)^{-1} = \dfrac{1}{\sin x} = \csc x$.

Some Important Properties of the Inverse of Trigonometric Functions

بعض الخواص المهمة لمعكوس الدوال المثلثية

(1) $\sin^{-1}(-x) = -\sin^{-1}x$.

(2) $\cos^{-1}(-x) = \pi - \cos^{-1}x$.

(3) $\tan^{-1}(-x) = -\tan^{-1}x$.

(4) $\cot^{-1}(-x) = -\cot^{-1}x$.

(5) $\sec^{-1}(-x) = \pi - \sec^{-1}x$.

(6) $\csc^{-1}(-x) = -\csc^{-1}x$.

(7) $\sin^{-1}x = \frac{\pi}{2} - \cos^{-1}x$.

(8) $\tan^{-1}x = \frac{\pi}{2} - \cot^{-1}x$.

(9) $\sec^{-1}x = \frac{\pi}{2} - \csc^{-1}x$.

(10) $\sin^{-1}x = \csc^{-1}\frac{1}{x}$.

(11) $\cos^{-1}x = \sec^{-1}\frac{1}{x}$.

(12) $\tan^{-1}x = \cot^{-1}\frac{1}{x}$.

(13) $\sin^{-1}(\sin x) = x$.

(14) $\sin(\sin^{-1}x) = x$.

Proof (1)

Let $y = \sin^{-1}(-x) \Rightarrow -x = \sin y \Rightarrow x = -\sin y = \sin(-y)$

$\Rightarrow -y = \sin^{-1}x \Rightarrow y = -\sin^{-1}x \Rightarrow \sin^{-1}(-x) = -\sin^{-1}x$.

Proof (2)

Let $y = \cos^{-1}(-x) \Rightarrow -x = \cos y \Rightarrow x = -\cos y \Rightarrow$

$x = \cos(\pi - y) \Rightarrow \pi - y = \cos^{-1}x \Rightarrow y = \pi - \cos^{-1}x$.

proof (3)

Let $y = \tan^{-1}(-x) \Rightarrow -x = \tan y \Rightarrow x = -\tan y$

$\Rightarrow x = \tan(-y) \Rightarrow -y = \tan^{-1} x \Rightarrow y = -\tan^{-1} x$.

proof (7)

Let $y = \sin^{-1} x \Rightarrow x = \sin y \Rightarrow x = \cos(\frac{\pi}{2} - y) \Rightarrow \frac{\pi}{2} - y = \cos^{-1} x$

$\Rightarrow y = \frac{\pi}{2} - \cos^{-1} x$

proof (10)

Let $y = \sin^{-1} x \Rightarrow x = \sin y \Rightarrow x = \frac{1}{\csc y} \Rightarrow \frac{1}{x} = \csc y$

$\Rightarrow y = \csc^{-1}(\frac{1}{x})$.

proof (13)

Let $y = \sin^{-1}(\sin x) \Rightarrow \sin y = \sin x \Rightarrow x = y$.

EX.(1) If $\theta = \sin^{-1}\frac{\sqrt{3}}{2}$. Find $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$ and $\csc\theta$.

solu.

$\theta = \sin^{-1}\frac{\sqrt{3}}{2} \Rightarrow \frac{\sqrt{3}}{2} = \sin\theta \Rightarrow \theta = \frac{\pi}{3}$

$\therefore \cos\frac{\pi}{3} = \cos\theta = \frac{1}{2}$ and so on ...

EX.(2) Evaluate

(1) $\sin(\cos^{-1}\frac{1}{\sqrt{2}})$ (2) $\sec(\cos^{-1}\frac{1}{2})$ (3) $\tan(\sin^{-1}(-\frac{1}{2}))$

(4) $\sin(\sin^{-1}\frac{\pi}{3})$.

solu.

1. Let $\theta = \cos^{-1}\frac{1}{\sqrt{2}} \Rightarrow \frac{1}{\sqrt{2}} = \cos\theta \Rightarrow \theta = \frac{\pi}{4}$

So $\sin(\cos^{-1}\frac{1}{\sqrt{2}}) = \sin\theta = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

2. Let $\theta = \cos^{-1}\frac{1}{2} \Rightarrow \frac{1}{2} = \cos\theta \Rightarrow \theta = \frac{\pi}{3}$

So $\sec(\cos^{-1}\frac{1}{2}) = \sec(\theta) = \sec(\frac{\pi}{3}) = 2$.

3. Let $\theta = \sin^{-1}(-\frac{1}{2}) \Rightarrow \sin\theta = -\frac{1}{2} \Rightarrow \theta = -\frac{\pi}{6}$

So $\tan(\sin^{-1}(-\frac{1}{2})) = \tan(\theta) = \tan(-\frac{\pi}{6}) = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$.

4. $\sin(\sin^{-1}\frac{\pi}{3}) = \frac{\pi}{3}$.

EX.(3) Solve for x if $\tan^{-1}x - \cot^{-1}x = \frac{\pi}{4}$

Solu.

$$\tan^{-1}x - (\frac{\pi}{2} - \tan^{-1}x) = \frac{\pi}{4} \Rightarrow \tan^{-1}x - \frac{\pi}{2} + \tan^{-1}x = \frac{\pi}{4}$$

$$\Rightarrow 2\tan^{-1}x = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$$

$$\Rightarrow \tan^{-1}x = \frac{3\pi}{8} \Rightarrow x = \tan\frac{3\pi}{8}$$

$$x = \tan\frac{3\pi}{8} = \tan\left(\frac{\frac{3\pi}{4}}{2}\right) = \frac{\sin\frac{3\pi}{4}}{1+\cos\frac{3\pi}{4}} = \frac{\sin(\frac{\pi}{2}+\frac{\pi}{4})}{1+\cos(\frac{\pi}{2}+\frac{\pi}{4})}$$

$$= \frac{\cos\frac{\pi}{4}}{1-\sin\frac{\pi}{4}} = \frac{\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}-1}.$$

ملاحظة

$$\boxed{\tan\frac{x}{2} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}}$$

Derivative of The Inverse of Trigonometric Functions

مشتقة معكوس الدوال المثلثية

Theorem:

(1) If $y = \sin^{-1}x$ then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$.

(2) If $y = \cos^{-1}x$ then $\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$.

(3) If $y = \tan^{-1}x$ then $\frac{dy}{dx} = \frac{1}{1+x^2}$

(4) If $y = \cot^{-1}x$ then $\frac{dy}{dx} = \frac{-1}{1+x^2}$

(5) If $y = \sec^{-1}x$ then $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$.

(6) If $y = \csc^{-1} x$ then $\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}}$

<u>proof(1)</u> $y = \sin^{-1} x \Rightarrow x = \sin y \Rightarrow 1 = \cos y \frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

<u>proof(2)</u>

$y = \cos^{-1} x \Rightarrow x = \cos y \Rightarrow 1 = -\sin y \frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}.$$

<u>proof(3)</u> $y = \tan^{-1} x \Rightarrow x = \tan y \Rightarrow 1 = \sec^2 y \frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

Now, if $u = u(x)$ is a differentiable function of x and

1. $y = \sin^{-1} u$ then $\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$.
2. $y = \cos^{-1} u$ then $\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx}$.
3. $y = \tan^{-1} u$ then $\frac{dy}{dx} = \frac{1}{1+u^2} \frac{du}{dx}$.
4. $y = \cot^{-1} u$ then $\frac{dy}{dx} = \frac{-1}{1+u^2} \frac{du}{dx}$.
5. $y = \sec^{-1} u$ then $\frac{dy}{dx} = \frac{1}{u\sqrt{u^2-1}} \frac{du}{dx}$.
6. $y = \csc^{-1} u$ then $\frac{dy}{dx} = \frac{-1}{u\sqrt{u^2-1}} \frac{du}{dx}$.

EX.(4) Find $\frac{dy}{dx}$ of the following functions:

(1) $y = \sin^{-1}(x^2+3x-1) \Rightarrow \frac{dy}{dx} = \frac{(2x+3)}{\sqrt{1-(x^2+3x-1)^2}}$.

(2) $y = x^2 \tan^{-1}\sqrt{x} \Rightarrow \frac{dy}{dx} = x^2 \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}} + 2x \tan^{-1}\sqrt{x}$.

(3) $y = \cos^{-1}(x^2+\tan^{-1}3x) \Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-(x^2+\tan^{-1}3x)}} \cdot \left(2x+\frac{3}{1+9x^2}\right)$

(4) $y = \sin^2(\sec^{-1}2x) \cdot \cot^{-1}\left(\frac{1}{x}\right)$

$$\frac{dy}{dx} = \sin^2(\sec^{-1}2x) \cdot \frac{-1}{1+\left(\frac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2}\right) + \cot^{-1}\left(\frac{1}{x}\right) \cdot 2\sin(\sec^{-1}2x)$$

$$\cos(\sec^{-1}2x) \cdot \frac{2}{2x\sqrt{4x^2-1}}.$$

EX.(5) Find $\frac{d}{dx}\left[\frac{\tan^2(3x^2+1)}{\sin^{-1}(x^2-1)}\right]$

$$= \frac{\sin^{-1}(x^2-1) \cdot 2\tan(3x^2+1)\sec^2(3x^2+1)(6x) - \tan^2(3x^2+1)\frac{(2x)}{\sqrt{1-(x^2-1)^2}}}{\left[\sin^{-1}(x^2-1)\right]^2}$$

EX.(6) If $y = \sin^{-1}\left(\frac{x-1}{x+1}\right)$. Find $\frac{dy^3}{d\sec 2x}$

solu.

Let $u = y^3 \Rightarrow \frac{du}{dy} = 3y^2$

$v = \sec 2x \Rightarrow \frac{dv}{dx} = 2\sec 2x \tan 2x$

$$\frac{dy^3}{d\sec 2x} = \frac{du}{dv} = \frac{du}{dy} \cdot \frac{dy}{dx} \cdot \frac{dx}{dv} = 3y^2 \cdot \frac{\frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}}{\sqrt{1-\left(\frac{1-x}{1+x}\right)^2}} \cdot \frac{1}{2\sec 2x \tan 2}$$

EX.(7) If $y = \sin^{-1} t$ and $x = \cos^{-1} t$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $t = \frac{1}{2}$

Solu.

$$y = \sin^{-1} t \Rightarrow \frac{dy}{dt} = \frac{1}{\sqrt{1-t^2}} \quad , \quad x = \cos^{-1} t \Rightarrow \frac{dx}{dt} = \frac{-1}{\sqrt{1-t^2}}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -1 \quad \text{and} \quad \frac{d^2y}{dx^2} = 0 .$$

H.W

EX.(8) Show that the functions

$$f(x) = \sin^{-1}\left(\frac{x-1}{x+1}\right) \text{ and } g(x) = 2\tan^{-1}\sqrt{x}$$

have the same derivative.

EX.(9): Evaluate the following Limits:

(1) $$\lim_{x \to 0} \frac{\sin^{-1} x}{x} = \lim_{x \to 0} \frac{1}{\sqrt{1-x^2}} = 1 .$$

(2) $$\lim_{x \to 0} \frac{2\tan^{-1} 3x}{5x} = \lim_{x \to 0} \frac{2\,\frac{3}{1+(3x)^2}}{5} = \frac{6}{5} .$$

(3) $$\lim_{x \to 0} \frac{\tan^{-1} x - x}{x^3} = \lim_{x \to 0} \frac{\frac{1}{1+x^2} - 1}{3x^2} = \lim_{x \to 0} \frac{\frac{(1+x^2)(0) - 1(2x)}{(1+x^2)^2}}{6x}$$

$$= \lim_{x \to 0} \frac{\frac{-2x}{(1+x^2)^2}}{6x} = \lim_{x \to 0} \frac{\frac{(1+x^2)^2(-2) - (-2x)\,2(1+x^2)(2x)}{(1+x^2)^4}}{6}$$

$$= \frac{\frac{-2-0}{1}}{6} = \frac{-2}{6} = -\frac{1}{3} .$$

H.W

(4) $$\lim_{x \to 0} \frac{\sin^{-1} x - x}{x^3} = \frac{1}{6}$$ استعمال طريقة ارشال ثلاث مرات .

The Logarithmic Function الدالة اللوغاريتمية

The logarithm was discovered by a Scottish Nobleman John Napier (1550 – 1617)

$$y = f(x) = \log_b x \iff x = b^y$$

where y is the logarithm
x is the number
b is the base

If $b = 10$, we write $y = \log_{10} x$ or $y = \log x$

If $b = e = 2.7183$, we write $y = \log_e x$ or $y = \ln x$

ln read natural logarithm (اللوغاريتم الطبيعي)

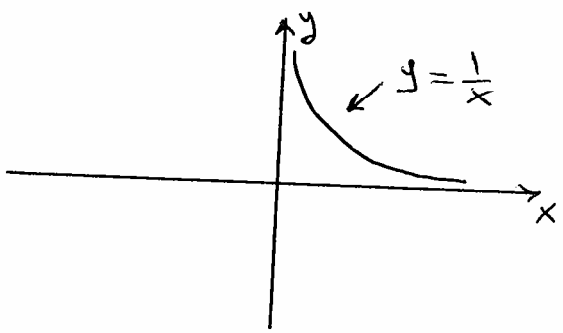
Relation Between The Logarithm and The Natural Logarithm

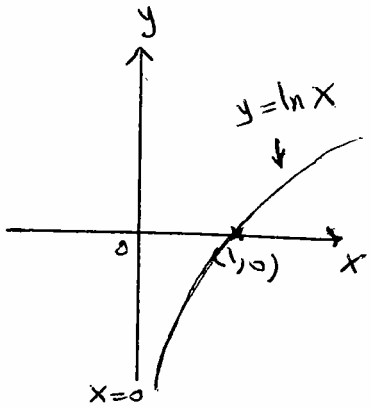
Let $y = \log_b x \iff x = b^y \Rightarrow \ln x = \ln b^y = y \ln b$

$$\Rightarrow y = \frac{\ln x}{\ln b}$$

So $$\log_b x = \frac{\ln x}{\ln b}$$

<u>Defn.</u>: For $x > 0$, we define $\ln x = \int_1^x \frac{dt}{t}$





Properties

1. $\ln(a \cdot b) = \ln a + \ln b$

2. $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

3. $\ln 1 = 0$

4. $\ln a^r = r \ln a$ where $r = \frac{p}{q}$, p and $q \neq 0$ are integers.

proof (1)

$$\ln(a.b) = \int_1^{a.b} \frac{dt}{t} = \int_1^{a} \frac{dt}{t} + \int_a^{a.b} \frac{dt}{t} = \ln a + \int_a^{a.b} \frac{dt}{t}$$

Let $u = \frac{t}{a} \Rightarrow au = t \Rightarrow a\,du = dt$

$$\int_a^{a.b} \frac{dt}{t} = \int_1^{b} \frac{a\,du}{at} = \int_1^{b} \frac{du}{u} = \ln(b)$$

$$\therefore \ln(a.b) = \ln a + \ln b$$

proof (2)

$$a = \frac{a}{b} \cdot b \Rightarrow \ln(a) = \ln\left(\frac{a}{b} \cdot b\right) = \ln\left(\frac{a}{b}\right) + \ln(b)$$

$$\Rightarrow \ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b).$$

proof (3)

$$1 = \frac{a}{a} \Rightarrow \ln 1 = \ln\left(\frac{a}{a}\right) = \ln a - \ln a = 0$$

proof (4) Let $u = a^{\frac{1}{q}}$ then $a^r = a^{\frac{p}{q}} = \left(a^{\frac{1}{q}}\right)^p = u^p$.

Also $a^r = u^p \Rightarrow \ln a^r = \ln u^p = p \ln u$ ~~(1)

$u = a^{\frac{1}{q}} \Rightarrow a = u^q \Rightarrow \ln a = q \ln u \Rightarrow \frac{1}{q} \ln a = \ln u$ ~~(2)

From (1) and (2)

$$\ln a^r = p \cdot \frac{1}{q} \ln a = \frac{p}{q} \ln a = r \ln a.$$

Derivative of The Natural Logarithm مشتقة دالة اللوغاريتم الطبيعي

Theorem

If $y = f(x) = \ln x$ then $\frac{dy}{dx} = f'(x) = \frac{1}{x}$

proof:

$$\frac{dy}{dx} = \frac{d}{dx}(\ln x) = \frac{d}{dx}\int_1^x \frac{dt}{t} = \frac{1}{x}$$ (by Fundamental Theorem of Calculus)

Now, If $u = u(x)$ is a differential function of x and

$$y = \ln u \quad \text{then} \quad \frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx}$$

EX.(1) Find $\frac{dy}{dx}$ of the following Functions:

(1) $y = \ln(x^3 + 2x^2 - 3x + 5) \Rightarrow \frac{dy}{dx} = \frac{3x^2 + 4x - 3}{(x^3 + 2x^2 - 3x + 5)}$

(2) $y = \ln(x^{-2} + \sin^2 3x) \Rightarrow \frac{dy}{dx} = \frac{(-2x^{-3} + 6\sin 3x \cos 3x)}{(x^{-2} + \sin^2 3x)}$

(3) $y = \sin^{-1}(\ln x) \cdot \ln(\sin^{-1} 3x)$

$$\frac{dy}{dx} = \sin^{-1}(\ln x) \cdot \frac{\frac{3}{\sqrt{1 - (3x)^2}}}{\sin^{-1} 3x} + \ln(\sin^{-1} 3x) \cdot \frac{\frac{1}{x}}{\sqrt{1 - (\ln x)^2}}$$

(4) $y = \ln[\ln(\sec^2 2x + x \sin^{-1} x)]$

$$\frac{dy}{dx} = \frac{1}{\ln(\sec^2 2x + x \sin^{-1} x)} \cdot \frac{1}{\sec^2 2x + x \sin^{-1} x} \cdot$$

$$\left(2\sec 2x \cdot \sec 2x \tan 2x \cdot 2 + \frac{x}{\sqrt{1 - x^2}} + \sin^{-1} x\right).$$

H.W

(5) $y = \frac{x + \ln(\sec 3x)}{\ln(x + \frac{1}{x})}$.

EX.(2) If $y = \dfrac{x^{3/2}(3x+2)^{1/2}(x^2+3x-1)^{4/3}}{(x^3+2)^{5/2}}$. Find $\dfrac{dy}{dx}$

Solu.

$$\ln y = \frac{3}{2}\ln x + \frac{1}{2}\ln(3x+2) + \frac{4}{3}\ln(x^2+3x-1) - \frac{5}{2}\ln(x^3+2)$$

$$\frac{1}{y}\frac{dy}{dx} = \left[\frac{3}{2}\frac{1}{x} + \frac{1}{2}\frac{3}{(3x+2)} + \frac{4}{3}\frac{(2x+3)}{(x^2+3x-1)} - \frac{5}{2}\frac{3x^2}{(x^3+2)}\right]$$

$$\therefore \frac{dy}{dx} = y\left[\ \text{----}\ \right]$$

EX.(3) If $y = \dfrac{(x^2+1)^{3/2}(x^2-1)^{-3/2}\tan^{-1}(\sin 2x)}{(x^2+4)^{2/3}\sin^{-3}2x}$. Find $\dfrac{dy}{dx}$

Solu.

$$\ln y = \frac{3}{2}\ln(x^2+1) - \frac{3}{2}\ln(x^2-1) + \ln[\tan^{-1}(\sin 2x)] - \frac{2}{3}\ln(x^2+4)$$
$$+3\ln(\sin 2x)$$

$$\frac{1}{y}y' = \left[\frac{3}{2}\frac{2x}{(x^2+1)} - \frac{3}{2}\frac{2x}{(x^2-1)} + \frac{\frac{2\cos 2x}{1+\sin^2 2x}}{\tan^{-1}(\sin 2x)} - \frac{2}{3}\frac{2x}{(x^2+4)}\right.$$
$$\left. + 6\frac{\cos 2x}{\sin 2x}\right]$$

$$\therefore y' = y\left[\ \text{-----}\ \right]$$

EX.(4) Find $\dfrac{dy}{dx}$ of the following functions:

① $y = x^{\sin x}$

solu. $\ln y = \sin x \ln x \Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{\sin x}{x} + \cos x \ln x$

$$\frac{dy}{dx} = y\left[\frac{\sin x}{x} + \cos x \cdot \ln x\right] = x^{\sin x}\left[\frac{\sin x}{x} + \cos x \cdot \ln x\right].$$

② $y = (\ln x)^x \Rightarrow \ln y = x \ln(\ln x)$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x \frac{1}{\ln x}\cdot\frac{1}{x} + \ln(\ln x)\cdot(1)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{\ln x} + \ln(\ln x)\right] = (\ln x)^x\left[\frac{1}{\ln x} + \ln(\ln x)\right].$$

③ $y = (\tan^{-1}x)^{\frac{x\sin x}{x^2+1}} \Rightarrow \ln y = \frac{x\sin x}{x^2+1}\cdot\ln(\tan^{-1}x)$

$$\Rightarrow \ln(\ln y) = \ln x + \ln\sin x - \ln(x^2+1) + \ln[\ln(\tan^{-1}x)]$$

$$\frac{1}{\ln y}\cdot\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{\cos x}{\sin x} - \frac{2x}{(x^2+1)} + \frac{1}{\ln(\tan^{-1}x)}\cdot\frac{1}{1+x^2}$$

$$\frac{dy}{dx} = y\ln y\left[\frac{1}{x} + \cot x - \frac{2x}{(x^2+1)} + \frac{1}{(1+x^2)\ln(\tan^{-1}x)}\right]$$

EX.(5) Evaluate the following Limits:

(1) $\lim\limits_{x\to\infty}\frac{\ln x}{x} = \lim\limits_{x\to\infty}\frac{\frac{1}{x}}{1} = \lim\limits_{x\to\infty}\frac{1}{x} = \frac{1}{\infty} = 0.$

(2) $\lim\limits_{x\to 0}\frac{\ln(1+2x)-2x}{x^2} = \lim\limits_{x\to 0}\frac{\frac{2}{1+2x}-2}{2x} = \lim\limits_{x\to 0}\frac{-\frac{4}{(1+2x)^2}}{2} = -2$

(3) $\lim\limits_{x\to\infty}\frac{\ln(\ln x)}{\ln x} = \lim\limits_{x\to\infty}\frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{\frac{1}{x}} = \lim\limits_{x\to\infty}\frac{1}{\ln x} = 0.$

EX.(6) Solve for x if $3^x = 2^{x+1}$

Solu. Take Logarithm of both side

$$x\ln 3 = (x+1)\ln 2 \Rightarrow x\ln 3 = x\ln 2 + \ln 2$$

$$x\ln 3 - x\ln 2 = \ln 2 \Rightarrow x(\ln 3 - \ln 2) = \ln 2$$

$$\therefore x = \frac{\ln 2}{\ln 3 - \ln 2}.$$

The Exponential function الدالة الاسية

Defn. The exponential function is defined as an inverse of the natural logarithm, and denoted by exp or e.

That is

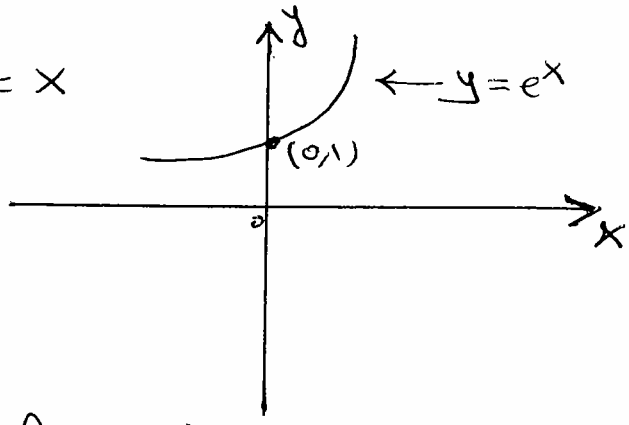
For $-\infty < x < \infty$, we define $y = f(x) = e^x \iff x = \ln y$

, $0 < y < \infty$.

properties

(1) $e = 2.7183$ (2) $e^{x+y} = e^x \cdot e^y$ (3) $e^{x-y} = \frac{e^x}{e^y}$

(4) $e^{\ln x} = x$ (5) $\ln e^x = x$



proof (2)

Let $u = e^x \Rightarrow x = \ln u$

Let $v = e^y \Rightarrow y = \ln v$

$x + y = \ln u + \ln v \Rightarrow x + y = \ln(u \cdot v)$

$\Rightarrow u \cdot v = e^{x+y} \Rightarrow e^x \cdot e^y = e^{x+y}$.

proof (4)

Let $y = e^{\ln x} \Rightarrow \ln x = \ln y \Rightarrow x = y = e^{\ln x}$.

proof (5)

Let $y = \ln e^x \Rightarrow e^y = e^x \Rightarrow x = y = \ln e^x$.

EX.(1) Simplify the following expressions :

بسط التعابير الآتية :

1. $e^{\ln 2} = 2$ 2. $e^{\ln(x^2+1)} = x^2 + 1$

3. $\ln e^{-1.3} = -1.3$ 4. $\ln e^{\sin x} = \sin x$

5. $\ln\left(\frac{e^{2x}}{5}\right) = \ln e^{2x} - \ln 5 = 2x - \ln 5$.

6. $e^{\ln 2 + 3\ln x} = e^{\ln 2} \cdot e^{3\ln x} = 2 \cdot e^{\ln x^3} = 2x^3$.

7. $e^{2x + \ln x} = e^{2x} \cdot e^{\ln x} = x e^{2x}$.

EX.(2) Solve for y if

(1) $\ln(y-1) - \ln y = 2x$

Solu. $\ln(y-1) - \ln y = 2x \Rightarrow \ln\left(\frac{y-1}{y}\right) = 2x$

$\Rightarrow \frac{y-1}{y} = e^{2x} \Rightarrow y - 1 = y e^{2x} \Rightarrow y(1 - e^{2x}) = 1$

$\therefore y = \frac{1}{1 - e^{2x}}$

(2) $\ln(y-1) = x \ln x \Rightarrow y - 1 = e^{x \ln x}$

$= e^{x} \cdot e^{\ln x} = x e^{x}$

$\Rightarrow y = x e^{x} + 1$.

Derivative of The Exponential Function مشتقة الدالة الأسية

Theorem

If $y = e^{x}$ then $\frac{dy}{dx} = e^{x}$

Proof

$y = e^{x} \Rightarrow x = \ln y \Rightarrow 1 = \frac{1}{y}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = y = e^{x}$.

Now, if $u(x) = u$ is a differentiable function of x and $y = e^{u}$ then $\frac{dy}{dx} = e^{u} \cdot \frac{du}{dx}$

EX.(1) Find $\frac{dy}{dx}$ of the following functions:

(1) $y = e^{x^2 + \sin 2x} \Rightarrow \frac{dy}{dx} = e^{x^2 + \sin 2x} \cdot (2x + 2\cos 2x)$.

(2) $y = e^{\tan^{-1}2x + \ln x} = x\, e^{\tan^{-1}2x} \Rightarrow \frac{dy}{dx} = x\, e^{\tan^{-1}2x} \cdot \frac{2}{1+(2x)^2} + e^{\tan^{-1}2x} \cdot (1)$

$\therefore \frac{dy}{dx} = \left(\frac{2x}{1+4x^2} + 1\right) e^{\tan^{-1}2x}$.

(3) $y = \tan^{-1}(e^{2x}) \Rightarrow \frac{dy}{dx} = \frac{2e^{2x}}{1+e^{4x}}$

(4) $y = e^{\sec x} \cdot \sec e^x \Rightarrow \frac{dy}{dx} = e^{\sec x} \cdot \sec(e^x) \cdot \tan(e^x)\, e^x$

$+ \sec(e^x) \cdot e^{\sec x} \sec x \tan x$

<u>EX(2)</u> If $y = (\sin x)^{e^x}$. Find $\frac{d\, e^{\sin^{-1}y}}{d \ln x}$

<u>solu.</u>

Let $u = e^{\sin^{-1}y}$ and $v = \ln x$

$$\frac{d\, e^{\sin^{-1}y}}{d \ln x} = \frac{du}{dv} = \frac{du}{dy} \cdot \frac{dy}{dx} \cdot \frac{dx}{dv}$$

$u = e^{\sin^{-1}y} \Rightarrow \frac{du}{dy} = e^{\sin^{-1}y} \cdot \frac{1}{\sqrt{1-y^2}}$, $v = \ln x \Rightarrow \frac{dv}{dx} = \frac{1}{x}$

$y = (\sin x)^{e^x} \Rightarrow \ln y = e^x \ln(\sin x)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \left[e^x \cdot \frac{\cos x}{\sin x} + \ln(\sin x)\, e^x\right]$

$\therefore \frac{dy}{dx} = y\left[e^x \cot x + e^x \ln(\sin x)\right]$

$\therefore \frac{dy}{dx} = (\sin x)^{e^x}\left[e^x \cot x + e^x \ln(\sin x)\right]$.

$\Rightarrow \frac{d\, e^{\sin^{-1}y}}{d \ln x} = \frac{e^{\sin^{-1}y}}{\sqrt{1-y^2}} \cdot (\sin x)^{e^x}\left[e^x \cot x + e^x \ln(\sin x)\right] \cdot x$.

where $y = (\sin x)^{e^x}$.

EX.(3) Evaluate the following Limits :

(1) $\lim\limits_{x\to\infty} \frac{e^x}{x^3} = \lim\limits_{x\to\infty} \frac{e^x}{3x^2} = \lim\limits_{x\to\infty} \frac{e^x}{6x} = \lim\limits_{x\to\infty} \frac{e^x}{6}$

$= \frac{e^\infty}{6} = \infty$.

(2) $\lim\limits_{x\to\infty} \frac{x^3}{e^x} = \lim\limits_{x\to\infty} \frac{3x^2}{e^x} = \lim\limits_{x\to\infty} \frac{6x}{e^x} = \lim\limits_{x\to\infty} \frac{6}{e^x} = \frac{6}{e^\infty} = \frac{6}{\infty} = 0$

(3) $\lim\limits_{x\to 0} \frac{e^{2x}-2x-1}{1-\cos x} = \lim\limits_{x\to 0} \frac{2e^{2x}-2}{\sin x} = \lim\limits_{x\to 0} \frac{4e^{2x}}{\cos x} = \frac{4e^0}{\cos 0} = \frac{4}{1} = 4$

(4) $\lim\limits_{x\to 0^+} x^{\frac{1}{x}} = \lim\limits_{x\to 0^+} e^{\ln x^{\frac{1}{x}}} = \lim\limits_{x\to 0^+} e^{\frac{1}{x}\ln x} = \lim\limits_{x\to 0^+} e^{\frac{\ln x}{x}}$

$= \lim\limits_{x\to 0^+} e^{\frac{\frac{1}{x}}{1}} = e^{\frac{\infty}{1}} = e^\infty = \infty$

(5) $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = \lim\limits_{x\to 0} e^{\ln(1+x)^{\frac{1}{x}}} = \lim\limits_{x\to 0} e^{\frac{\ln(1+x)}{x}} = \lim\limits_{x\to 0} e^{\frac{1}{1+x}}$

$= e^{\frac{1}{1+0}} = e^1 = e$.

H.W

(6) $\lim\limits_{x\to\infty} \left(1+\frac{b}{x}\right)^{cx} = e^{bc}$.

H.W

EX.(4) Solve for x if $3y = \frac{e^{2x}-1}{e^{2x}+1}$

H.W

EX.(5) $\lim\limits_{x\to 4} \frac{e^{x-4}-x+4}{\cos^2(\pi x)}$.

The Function a^x

Defn. For $a > 0$, we define $a^x = e^{x \cdot \ln a}$

Theorem If $y = a^x$ then $\frac{dy}{dx} = a^x \cdot \ln a$

proof

$$y = a^x = e^{x \ln a} \Rightarrow \frac{dy}{dx} = e^{x \ln a} \cdot \ln a$$
$$= a^x \cdot \ln a .$$

Now, if $u = u(x)$ is a differentiable function of x and

$$y = a^u \text{ then } \frac{dy}{dx} = a^u \cdot \ln a \cdot \frac{du}{dx}$$

EX.(1) Find $\frac{dy}{dx}$ of the following functions:

① $y = 2^{\sin^2 2x} \Rightarrow \frac{dy}{dx} = 2^{\sin^2 2x} \cdot \ln 2 \cdot (2 \sin 2x \cos 2x \cdot 2)$

$$= 2^{\sin^2 2x} \cdot \ln 2 \cdot (4 \sin 2x \cos 2x)$$

② $y = 3^{\tan^{-1} 2x} \cdot \ln(\sec 2x)$.

$$y' = \frac{dy}{dx} = 3^{\tan^{-1} 2x} \cdot \frac{2 \sec 2x \cdot \tan 2x}{\sec 2x} + \ln(\sec 2x) \cdot 3^{\tan^{-1} 2x} \cdot \ln 3 \left(\frac{2}{1+4x^2} \right)$$

$$= 2 \cdot 3^{\tan^{-1} 2x} \cdot \tan(2x) + \ln 3 \cdot \ln(\sec 2x) \cdot 3^{\tan^{-1} 2x} \cdot \frac{2}{1+4x^2} .$$

EX.(2) Find the following Limits:

(1) $\lim_{x \to \infty} 2^{-x} = 2^{-\infty} = 0$ (2) $\lim_{x \to -\infty} 3^x = 3^{-\infty} = 0$.

(3) $\lim_{x \to 0} \frac{3^{\sin x} - 1}{x} = \lim_{x \to 0} \frac{3^{\sin x} \ln 3 \cos x}{1} = 3^0 \cdot \ln 3 \cdot \cos 0 = (1) \cdot \ln 3 \cdot (1)$
$= \ln 3$.

4. $$\lim_{x\to 0} \frac{3^x - 2^x}{x} = \lim_{x\to 0} \frac{3^x \ln 3 - 2^x \ln 2}{1}$$

$$= 3^0 \ln 3 - 2^0 \ln 2 = \ln 3 - \ln 2 = \ln\left(\frac{3}{2}\right).$$

EX. (3) Solve for x if $3^{\log_3 7} + 2^{\log_2 5} = 5^{\log_5 x}$

Solu.

Since $3^{\log_3 7} = e^{\log_3 7 \cdot \ln 3} = e^{\frac{\ln 7}{\ln 3}\cdot \ln 3} = e^{\ln 7} = 7.$

And $2^{\log_2 5} = e^{\log_2 5 \cdot \ln 2} = e^{\frac{\ln 5}{\ln 2}\cdot \ln 2} = e^{\ln 5} = 5$

So $7 + 5 = x \Rightarrow x = 12.$

The Hyperbolic functions الدوال الزائدية

The hyperbolic functions are a special combinations of the functions e^x and e^{-x}.

تراكيب خاصة

الدوال الزائدية تراكيب خاصة من الدوال e^x و e^{-x}.

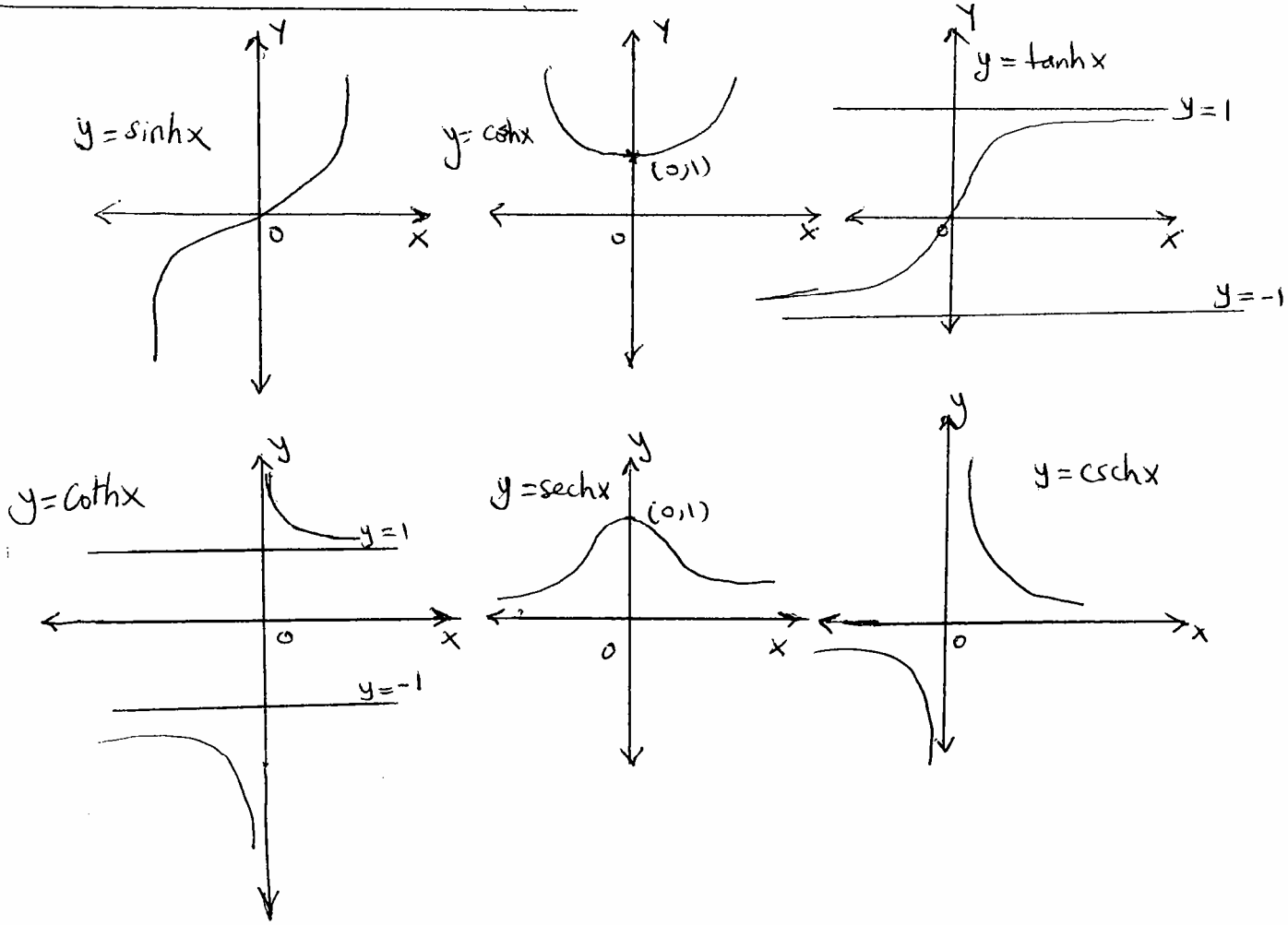
Defn. We define,

(1) $\sinh x = \dfrac{e^x - e^{-x}}{2}$ (2) $\cosh x = \dfrac{e^x + e^{-x}}{2}$

(3) $\tanh = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ (4) $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}} = \dfrac{1}{\tanh x}$

(5) $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$ (6) $\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

Graph of Hyperbolic Functions



Some Important Relations And Identities بعض العلاقات والمتطابقات المهمة

(1) $\cosh^2 x - \sinh^2 x = 1$. (2) $\tanh^2 x + \operatorname{sech}^2 x = 1$. (3) $\coth^2 x - \operatorname{csch}^2 x = 1$.

(4) $\sinh(-x) = -\sinh x$. (5) $\cosh(-x) = \cosh x$. (6) $\tanh(-x) = -\tanh$

(7) $\sinh x \pm \cosh x = \pm e^{\pm x}$.

(8) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$.

(9) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$.

(10) $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$.

(11) $\sinh 2x = 2 \sinh x \cosh x$.

(12) $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$. (13) $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$.

Proof(1): $\cosh^2 x - \sinh^2 x = \left(\frac{e^{x}+e^{-x}}{2}\right)^2 - \left(\frac{e^{x}-e^{-x}}{2}\right)^2$

$$= \frac{e^{2x}+2+e^{-2x}}{4} - \frac{e^{2x}-2+e^{-2x}}{4} = \frac{\cancel{e^{2x}}+2+\cancel{e^{-2x}}-\cancel{e^{2x}}+2-\cancel{e^{-2x}}}{4} = \frac{4}{4} = 1$$

Proof(2)

$$\tanh^2 x + \operatorname{sech}^2 x = \frac{\sinh^2 x}{\cosh^2 x} + \frac{1}{\cosh^2 x} = \frac{\sinh^2 x + 1}{\cosh^2 x} = \frac{\cosh^2 x}{\cosh^2 x} = 1.$$

proof(4) $\sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = -\frac{e^{x}-e^{-x}}{2} = -\sinh x.$

proof(5) $\cosh(-x) = \frac{e^{-x}+e^{-(-x)}}{2} = \frac{e^{x}+e^{-x}}{2} = \cosh x.$

proof(6) $\tanh(-x) = \frac{\sinh(-x)}{\cosh(-x)} = \frac{-\sinh x}{\cosh x} = -\tanh x.$

proof(7) $\sinh x + \cosh x = \frac{e^{x}-e^{-x}}{2} + \frac{e^{x}+e^{-x}}{2} = \frac{e^{x}-e^{-x}+e^{x}+e^{-x}}{2}$

$$= \frac{2e^{x}}{2} = e^{x}.$$

EX.(1) If $\sinh x = -\frac{3}{4}$. find the value of the other hyperbolics

Solu.

$$\cosh^2 x = 1 + \sinh^2 x = 1 + \left(\frac{-3}{4}\right)^2 = 1 + \frac{9}{16} = \frac{25}{16} \Rightarrow$$

Since $\cosh x > 0 \Rightarrow \cosh x = \frac{5}{4}$

$\tanh x = \frac{\sinh x}{\cosh x} = \frac{-\frac{3}{4}}{\frac{5}{4}} = \frac{-3}{4}$. and so on for the other hyperbolics.

H.W
EX.(2) Show that $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$.

H.W
EX.(3) If $\sinh x = \tan\theta$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Find the other hyperbolics in terms of the trigonometric.

H.W
EX.(4) Show that: $\sinh 3x = 3\sinh x + 4\sinh^3 x$.