Vector Analysis

Scalars and Vectors:

Quantities such as time, temperature and speed, which have manitude only, are called scalars, and are merely numbers.

Quantities such as force, velocity, acceleration and momentum, which have both magnitude and direction are called vectors. Vectors are represented geometrically by directed line segments (arrows). The direction of the arrow (the angle which it makes with some fixed line of the plane) is the direction of the vector, and the length of the arrow represent the magnitude of the vector. Vectors will be denoted by letters a, b, c, ... or as OP see figure (1). the magnitude of a vector OP will be denoted by |OP|

Figure (1)

Sum and Difference of two Vectors

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If a and b are the vectors of figure (1), their sum a+b is found by taking the vectors as in fig. (2) and completing the parallelogram PAQB. The vector PQ is the requried sum.

Fig. (2)

If a, b, c are vectors and k is a scalar, then

1. $a+b = b+a$ Commutative Law (قانون الإبدال)
2. $(a+b)+c = a+(b+c)$ Associative Law (قانون التجميع)
3. $k(a+b) = ka + kb$ Distributive Law (قانون التوزيع)

<u>A unit vector</u>: is a vector of length equal to 1

$|\vec{A}| = 1$ and written $\hat{A} = \dfrac{\vec{A}}{|A|}$

From Fig. (2) above all the vectors written as a and b and can be written as $\vec{A}$ and $\vec{B}$ and $\vec{C}$

if they have the same quality (Length) and direction they are equal.

<u>The rectangular unit Vectors i, j, k</u>

Component of a vector:

if $\vec{A} = A_1 i + A_2 j + A_3 k$ we shall call $A_1 i$, $A_2 j$, $A_3 k$ the vector components of A and

$$|A| = \sqrt{A_1^2 + A_2^2 + A_3^2}$$ (length or magnitude)

In particular the position vector

$$\vec{r} = x i + y j + z k$$

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2} \quad .$$

And the direction cosines of the position vector

$$\vec{r} = |r| \cos\alpha\, i + |r| \cos\beta\, j + |r| \cos\gamma\, k$$

where (α, β, γ direction cosines)

$$|\vec{r}| = \sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma} = 1$$

Examples: For vectors $a = 3i + 4j$ and $b = 2i - j$, find the magnitude and direction of

a and b, $a+b$, $a-b$

Solu. For $a = 3i + 4j$, $|a| = \sqrt{a_1^2 + a_2^2} = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$.

$\tan\theta = \frac{a_2}{a_1} = \frac{4}{3}$, $a_1 = |a| \cos\theta$ and $\cos\theta = \frac{3}{5}$. Then

$\theta = 53.8$.

For $b = 2i - j$, $|b| = \sqrt{b_1^2 + b_2^2} = \sqrt{4+1} = \sqrt{5}$

$\tan\theta = \frac{b_2}{b_1} = \frac{-1}{2}$, $\cos\theta = \frac{2}{\sqrt{5}} \Rightarrow \theta = 360 - 26.34 = 333.26$

For $a+b = (3i+4j)+(2i-j) = 5i+3j$

$|a+b| = \sqrt{5^2+3^2} = \sqrt{34}$, $\tan\theta = \frac{3}{5}$, $\cos\theta = \frac{5}{\sqrt{34}}$

$\theta = 30.58$

For $b-a = (2i-j)-(3i+4j) = -i-5j$

$|b-a| = \sqrt{26}$, $\tan\theta = 5$, $\cos\theta = \frac{-1}{\sqrt{26}}$

$\Rightarrow \theta = 258.41$

Product of two Vectors

حاصل ضرب متجهين

(a) Scalar product – dot product

The scalar or dot product of two vectors $\vec{A}$ and $\vec{B}$ is defined by

$$\vec{A}\cdot\vec{B} = |A||B|\cos\theta$$

From the equation follow:

$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ ابدالي

$\vec{A}\cdot\vec{A} = |A|\cdot|A| = |a|^2$

$\vec{A}\cdot\vec{B} = 0$ if $\vec{A}=0$ or $\vec{B}=0$ or $\vec{A} \perp$ to $\vec{B}$ عمودي

i.e $(\theta = 90^\circ \quad \cos\theta = 0)$

$i\cdot i = j\cdot j = 1 \qquad i\cdot j = 0$

$\vec{A}\cdot\vec{B} = (a_1 i + a_2 j)\cdot(b_1 i + b_2 j) = a_1 b_1 + a_2 b_2$

$\vec{A} \parallel \vec{B} \longrightarrow \vec{A}\cdot\vec{B} = |A||B|$ i.e $\theta = 0$

$\vec{A}\cdot\vec{B} = |A|$ projection of $\vec{B}$ on $\vec{A}$
$\quad\quad\quad = |B|$ projection of $\vec{A}$ on $\vec{B}$

b. In Cartesian Coordinates if

$$\vec{A} = a_1 i + a_2 j + a_3 k \quad , \quad \vec{B} = b_1 i + b_2 j + b_3 k$$

Then $\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$; where

$$i \cdot i = j \cdot j = k \cdot k = 1 \quad \text{and}$$
$$i \cdot j = j \cdot k = k \cdot i = 0$$

Now $A \cdot B = |A|\,|B| \cos\theta$

$$\cos\theta = \frac{A \cdot B}{|A|\,|B|}$$

$$\therefore \cos\theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\ \sqrt{b_1^2 + b_2^2 + b_3^2}}$$

<u>Example</u> : Find the <u>angle</u> between the vectors

$$\vec{A} = i + j + k \quad , \quad \vec{B} = -i + 2j + 3k$$

$$|A| = \sqrt{1+1+1} = \sqrt{3} \quad , \quad |B| = \sqrt{1+4+9} = \sqrt{14} .$$

$$\cos\theta = \frac{(1)(-1) + (1)(2) + (1)(3)}{\sqrt{3}\ \sqrt{14}} = \frac{-1+2+3}{\sqrt{42}} = \frac{4}{\sqrt{42}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{4}{\sqrt{42}}\right) .$$

<u>Example</u> : Find the value of m such that $\vec{A} = 2i + mj + k$ and $\vec{B} = 4i - 2j - 2k$ are perpendicular. متعامدة

<u>Solu.</u> if $\vec{A} \perp \vec{B} \Rightarrow \vec{A} \cdot \vec{B} = 0 \Rightarrow 8 - 2m - 2 = 0$

$$\Rightarrow 6 = 2m \Rightarrow m = 3$$

<u>Example</u>: Find the projection of $\vec{A}$ on $\vec{B}$, $\vec{B}$ on $\vec{A}$

where $\vec{A} = i + 2j + k$, $\vec{B} = 4i - 4j + 7k$

<u>Solu.</u> we have

$$\vec{A} \cdot \vec{B} = |A||B| \cos\theta$$

$$\text{proj. } \vec{B} \text{ on } \vec{A} = |B| \cos\theta = \frac{\vec{A} \cdot \vec{B}}{|A|} = \frac{4-8+7}{\sqrt{6}} = \sqrt{\frac{3}{2}}$$

$$\text{proj. } \vec{A} \text{ on } \vec{B} = |A| \cos\theta = \frac{\vec{A} \cdot \vec{B}}{|B|} = \frac{3}{\sqrt{81}} = \frac{1}{3}$$

<u>Example</u>: prove the law of cosines for a plane triangle

<u>Solu.</u> $\vec{B} + \vec{C} = \vec{A}$ <u>or</u>

$$\vec{C} = \vec{A} - B$$

$$\vec{C} \cdot \vec{C} = (\vec{A} - \vec{B})(\vec{A} - \vec{B})$$

$$\vec{C} \cdot \vec{C} = \vec{A} \cdot \vec{A} + \vec{B} \cdot \vec{B} - 2\vec{A}\vec{B}$$

$$|\vec{C}|^2 = |A|^2 + |B|^2 - 2|A||B| \cos\theta \quad \text{... Law of Cosine}$$

Cross product

Take $a = a_1 i + a_2 j + a_3 k = \vec{A}$

$b = b_1 i + b_2 j + b_3 k = \vec{B}$

with initial point denoted by **M** the unit vector normal to the plane of a and b ($\vec{A}$ and $\vec{B}$). see Fig. (3).

The vector or cross product of $\vec{A}$ and $\vec{B}$ is defined by

$$\vec{A} \times \vec{B} = |A||B| \sin\theta n \quad \text{—— (1)}$$

where n is a vector $\perp$ to both $\vec{A}$ and $\vec{B}$.

Geometrically:

$|A \times B| = |A||B| \sin\theta$ = the area of the parallegram having $\vec{A}$ and $\vec{B}$ as nonparallel sides

Fig.(3)

If $\vec{A}$ and $\vec{B}$ parallel, $\theta = 0$ or π

and $\vec{A} \times \vec{B} = 0$ thus

$i \times i = j \times j = k \times k = 0$

If in eq.(1) the order of $\vec{A}$ and $\vec{B}$ is reversed, n must be replaced by $-n$; hence

$$\vec{B} \times \vec{A} = -(\vec{A} \times \vec{B}).$$

Since the coordinate axis were chosen as a right-hand system, it follows that

$$i \times j = k \qquad j \times k = i \qquad k \times i = j$$
$$j \times i = -k \qquad k \times j = -i \qquad i \times k = -j$$

note that for any vectors $\vec{A}, \vec{B}, \vec{C}$, the distributive Law.

$$(\vec{A} + \vec{B}) \times \vec{C} = \vec{A} \times \vec{C} + \vec{B} \times \vec{C}.$$

There follow

$$(\vec{A} + \vec{B}) \times (\vec{C} + \vec{D}) = \vec{A} \times \vec{C} + \vec{A} \times \vec{D} + \vec{B} \times \vec{C} + \vec{B} \times \vec{D}$$

and

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

<u>Example</u>: If $\vec{A} = 2i - 3j - k$, $\vec{B} = i + 4j - 2k$

<u>solution</u>

$$A \times B = \begin{vmatrix} i & j & k \\ 2 & -3 & -1 \\ 1 & 4 & -2 \end{vmatrix} = 10i + 3j + 11k$$

$$B \times A = \begin{vmatrix} i & j & k \\ 1 & 4 & -2 \\ 2 & -3 & -1 \end{vmatrix} = -10i - 3j - 11k$$

the area of the parallelogram $= |A \times B| = \sqrt{100+9+121}$
$= \sqrt{230}$ square unit.

The product of Triple Scalar:

In Fig. (4), let θ be the smaller angle between B and C and let ϕ be the smaller angle between A and $B \times C$. Then the triple scalar product is by definition

$$A \cdot (B \times C) = A\,|B|\,|C| \sin\theta\, n = |A|\,|B|\,|C| \sin\theta \cos\phi$$
$$= (|A| \cos\phi)(|B|\,|C| \sin\theta) = nA$$
$$= \text{Volume of parallelopiped.}$$

it may be shown that

$$A \cdot (B \times C) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (A \times B) \cdot C$$

Fig. (4)

and from the properties of the determinantes

- $A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B)$
- $A \cdot (B \times C) = -B (A \times C)$.
- $A \cdot (B \times A) = 0$ and $B \cdot (B \times A) = 0$

- Geometrically

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{A} \cdot S\hat{n} \quad \text{where } S = \text{area}$$
$$= (\vec{A} \cdot \hat{n}) S$$
$$= S\,|A| \cos\theta = sh = V = \text{The Volume of the parallelopiped}$$

- if $A \cdot (B \times C) = 0$ the three vector are Coplanar.

Vector triple product

$$\vec{A} \times (\vec{B} \times \vec{C}) = (A \cdot C) B - (A \cdot B) C$$
$$(\vec{A} \times \vec{B}) \times \vec{C} = (A \cdot C) B - (B \cdot C) A$$

in case B is $\perp$ to both A and C

$$A \times (B \times C) \neq (A \times B) \times C$$

Example If $\vec{A} = 2i - 3j$, $\vec{B} = i + j - k$, $\vec{C} = 3i - k$

Find $\vec{A} \cdot (\vec{B} \times \vec{C})$

Solution

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} 2 & -3 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix} = 4$$

The straight line :

A line in space through a given point $P_0(x_0, y_0, z_0)$ will be defined as the locus of all points $p(x, y, z)$ such that P_0P is parallel to a given direction

$A = a_1 i + a_2 j + a_3 k$, let r_0, r be the position vector vectors of P_0 and P then

$$r - r_0 = ka \quad (k \text{ is a scalar Variable}) \quad \text{——— (1)}$$

is the vector equation of the line PP_0 , fig. (5)

written eq.(1) as

$$(x - x_0)i + (y - y_0)j + (z - z_0)k = k(a_1 i + a_2 j + a_3 k).$$

then

$$x - x_0 = ka_1$$
$$y - y_0 = ka_2$$
$$z - z_0 = ka_3$$

fig. (5)

eliminating K we have

$$\frac{x-x_0}{a_1} = \frac{y-y_0}{a_2} = \frac{z-z_0}{a_3} \qquad \text{——— (2)}$$

a_1, a_2, a_3 are a set of direction numbers for line and $\frac{a_1}{|A|}, \frac{a_2}{|A|}, \frac{a_3}{|A|}$ are a set of direction cosine of the line.

if $a_1 = 0$ $a_2 a_3 \neq 0$ the equations of the line are

$$x - x_0 = 0 \qquad \frac{y-y_0}{a_2} = \frac{z-z_0}{a_3}$$

The Plane

A plane is a space through a given point $P_0(x_0, y_0, z_0)$ will be defined as the locus of all lines through P_0 and perpendicular (normal) to a given line (direction) $a = Ai + Bj + Ck$.

Let $P(x, y, z)$ be any other point in the plane.

Then $r - r_0 = P_0P$ is $\perp$ to a, as in fig.(6) and required equation is

$$(r - r_0)\, a = 0$$

in rectangular coordinates, this becomes

$$[(x-x_0)i + (y-y_0)j + (z-z_0)k] \cdot (Ai + Bj + Ck) = 0$$

Fig.(6)

or $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$

$$Ax + By + Cz + D = 0$$

where $D = -(Ax_0 + By_0 + Cz_0)$.

Let $P_0(x_0, y_0, z_0)$ be a point on the surface

$$Ax + By + Cz + D = 0 \quad \text{——— (1)}$$
$$Ax_0 + By_0 + Cz_0 + D = 0 \quad \text{——— (2)}$$

Subtracting

$$A(x-x_0) + B(y-y_0) + C(z-z_0)$$
$$= (Ai + Bj + Ck) \cdot \{(x-x_0)i + (y-y_0)j + (z-z_0)k\} = 0$$

and the constant vector $Ai + Bj + Ck$ is normal to the surface at each of its points. Thus the surface is a plane.

<u>EX:</u> Find the angle α from the fig. shown whose vertices are $A(1,0,1)$, $B(2,-1,1)$, $C(-2,1,0)$

<u>Solu.</u>

$a = AC = -3i + j - k$

$b = AB = i - j$

$$\cos\alpha = \frac{a \cdot b}{|a||b|} = \frac{-3-1}{\sqrt{22}} = -0.852$$

$$\therefore \alpha = 148^\circ$$

<u>EX.</u> Find the direction cosine of $a = 3i + 12j + 4k$

<u>Solu.</u>

The direction cosine consists are

$$\cos\alpha = \frac{i \cdot a}{|a|} = \frac{3}{13} \qquad \cos\beta = \frac{j \cdot a}{|a|} = \frac{12}{13},$$

$$\cos\gamma = \frac{k \cdot a}{|a|} = \frac{4}{13}.$$

Vector-Valued functions

Consider functions for which the domain consists of real numbers and the range consists of vector in 2-space or 3-space. Such functions are called vector-valued functions of a real number (variable). In 2-space such functions can be expressed in the form

$$f(t) = \langle x(t), y(t) \rangle = x(t)i + y(t)j \quad \text{—— (1)}$$

and in 3-space in the form

$$f(t) = \langle x(t), y(t), z(t) \rangle = x(t)i + y(t)j + z(t)k \quad \text{—— (2)}$$

where $x(t), y(t), z(t)$ are called real-valued functions of the real variable t.

These real-valued functions are called component functions.

Ex.: If $f(t) = (\ln t)i + (\sqrt{t^2+2})j + [\cos(t\pi)]\,k$ then the components functions are

$$x(t) = \ln t \quad , \quad y(t) = \sqrt{t^2+2} \quad , \quad z(t) = \cos t\pi .$$

The vector that $f(t)$ associates with $t=1$ is

$$f(1) = (\ln 1)\,i + \sqrt{3}\,j + (\cos\pi)k = \sqrt{3}\,j - k .$$

Eq. (1) and (2) can be written by introducing vectors r and $f(t)$ defined by

$$r = xi + yj \quad , \quad f(t) = x(t)i + y(t)j$$

and $r = xi + yj + zk$, $f(t) = x(t)i + y(t)j + z(t)k$.

Ex. Express the given parametric equations as a single vector equation:

(a) $x = t^2$,
$y = 3t$.

(b) $x = \cos t$,
$y = \sin t$,
$z = t$.

Solu.

(a) $xi + yj = t^2 i + 3t j$

and $r = xi + yj$ and $f(t) = t^2 i + 3t j$

(b) $xi + yj + zk = (\cos t) i + (\sin t) j + t k$.

$r = xi + yj + zk$, $f(t) = (\cos t) i + (\sin t) j + t k$.

Vector Differentiation اشتقاق متجهة

Definition: The derivative $f'(t)$ of a vector-value function $f(t)$ is defined by

$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

Hence if $f(t) = x(t) i + y(t) j + z(t) k$ is a vector value function, then

$$f'(t) = x'(t) i + y'(t) j + z'(t) k.$$

Ex. find $f'(t)$, if $f(t) = (\cos t) i + (\sin t) j + t k$.

$$f'(t) = \frac{d}{dt}(\cos t) i + \frac{d}{dt}(\sin t) j + \frac{d}{dt}(t) k$$

$$= (-\sin t) i + (\cos t) j + k.$$

Properties of Derivatives:

1. $\frac{d}{dt}[k\, r(t)] = k \frac{d}{dt}[r(t)]$

2. $\frac{d}{dt}[f(t) r(t)] = f(t) \frac{d}{dt}[r(t)] + r(t) \frac{d}{dt}[f(t)]$

3. $\frac{d}{dt}[r_1(t) \cdot r_2(t)] = r_1(t) \frac{dr_2}{dt} + \frac{dr_1}{dt} \cdot r_2(t)$.

4. $\frac{d}{dt}[r_1(t) \times r_2(t)] = r_1(t) \times \frac{dr_2}{dt} + \frac{dr_1}{dt} \times r_2(t)$.

Ex.: Calculate $\frac{d}{dt}[r_1(t) \cdot r_2(t)]$ if

$r_1(t) = 2t\,i + 3t^2 j + t^3 k$, $r_2(t) = t^4 k$

Solu.

$$\frac{d}{dt}[r_1(t) \cdot r_2(t)] = r_1(t) \cdot \frac{dr_2}{dt} + \frac{dr_1}{dt} \cdot r_2(t)$$

$$= (2t\,i + 3t^2 j + t^3 k) \cdot 4t^3 k + (2i + 6t\,j + 3t^2 k)\, t^4 k .$$

$$= 4t^6 + 3t^6 = 7t^6 .$$

Ex.: if $A = 5t^2 i + t\,j - t^3 k$

$B = \sin t\, i - \cos t\, j$ find

(a) $\frac{d}{dt}(A \cdot B)$ (b) $\frac{d}{dt}(A \times B)$ (c) $\frac{d}{dt}(A \cdot A)$.

Solu.

(a) $$\frac{d}{dt}(A \cdot B) = A \cdot \frac{dB}{dt} + \frac{dA}{dt} \cdot B$$

$$= (5t^2 i + t\,j - t^3 k) \cdot (\cos t\, i + \sin t\, j) + (10t\, i + j - 3t^2 k) \cdot (\sin t\, i - \cos t\, j)$$

$$= 5t^2 \cos t + t \sin t + 10t \sin t - \cos t$$

$$= (5t^2 - 1) \cos t + 11t \sin t .$$

(b) $$\frac{d}{dt}(A \times B) = A \times \frac{dB}{dt} + \frac{dA}{dt} \times B$$

$$= \begin{vmatrix} i & j & k \\ 5t^2 & t & -t^3 \\ \cos t & \sin t & 0 \end{vmatrix} + \begin{vmatrix} i & j & k \\ 10t & 1 & -3t^2 \\ \sin t & -\cos t & 0 \end{vmatrix}$$

$$= [i(t^3 \sin t) - j(t^3 \cos t) + k(5t^2 \sin t - t\cos t)]$$
$$+ [i(-3t^2 \cos t) - j(3t^2 \sin t) + k(-10t \cos t - \sin t)]$$

$$= (t^3 \sin t - 3t^2 \cos t)\, i - (t^3 \cos t + 3t^2 \sin t)\, j$$
$$+ (5t^2 \sin t - \sin t - 10t \cos t)\, k$$

(c) $\frac{d}{dt}(A \cdot A) = A \frac{dA}{dt} + \frac{dA}{dt} \cdot A = 2A \frac{dA}{dt}$

$$= 2(5t^2 i + t j - t^3 k) \cdot (10t\, i + j - 3t^2 k)$$
$$= 100t^3 + 2t + 6t^5 .$$

H.W

<u>EX.</u> An object on the curve whose equations are $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$

find its speed and acceleration at:

① at any time

② at $t = 0$.

<u>Solution</u>

① The position vector r of the object is

$$r = x i + y j + z k = e^{-t} i + 2\cos 3t\, j + 2\sin 3t\, k$$

The speed $= V = \frac{dr}{dt} = -e^{-t} i - 6\sin 3t\, j + 6\cos 3t\, k$

and the acceleration is $a = \frac{d^2 r}{dt^2}$

$$a = e^{-t} i - 18\cos 3t\, j - 18\sin 3t\, k$$

(at any time).

② at $t = 0$, $\left.\frac{dr}{dt}\right|_{t=0} = -i + 6k = V$

$$|V| = \sqrt{(-1)^2 + 6^2} = \sqrt{37} .$$

and $a\Big|_{t=0} = i - 18j \Rightarrow |a| = \sqrt{(1)^2 + (18)^2} = \sqrt{325}$.

Integration of Vector-Valued Functions

If $f(t) = x(t)\,i + y(t)\,j + z(t)\,k$

then $\int f(t)\,dt = \left(\int x(t)\,dt\right) i + \left(\int y(t)\,dt\right) j + \left(\int z(t)\,dt\right) k$

EX. if $f(t) = 2t\,i + 3t^2 j$. Find

(a) $\int f(t)\,dt$ (b) $\int_0^2 f(t)\,dt$

Solu.

(a) $\int f(t)\,dt = \int (2t\,i + 3t^2 j)\,dt = t^2 i + t^3 j + C$

where $C = C_1 i + C_2 j$

(b) $\int_0^2 f(t)\,dt = t^2\Big|_0^2 i + t^3\Big|_0^2 j = 4i + 8j$.

H.W

EX. ~~The acceleration~~

The acceleration of a particle at any times $t \geq 0$ is given by $a = \frac{dv}{dt} = e^t j + e^{3t} j + k$. if at $t=0$, the displacement is $r=0$ and the velocity is $v = i + j$. find r and v at any time t.

H.W

~~Ex. Solu.~~

EX. If the velocity $v = -e^{-t} i - 6\sin 3t\, j + 6\cos 3t\, k$

Find the distance travelled.

Scalar and Vector Field

If $\phi(x,y,z)$ is differentiable scalar function then we can define the gradient of ϕ (usually defined by (grad ϕ)) as:

$$\text{grad}\,\phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

Ex. Given $\phi(x,y,z) = xyz$

$$\text{grad}\,\phi = yz\, i + xz\, j + xy\, k.$$

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we can write $\text{grad}\,\phi = \nabla\phi = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)\phi$

$\nabla\phi$ is defined vector field.

$$\nabla \cdot \nabla \phi = \nabla^2 \phi$$

if $\nabla^2 \phi = 0$ then $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$

which is called Laplace differential equation in three dimensions.

Divergence

if $V(x,y,z)$ is a vector function then $V(x,y,z) = V_1 i + V_2 j + V_3 k$ and the div of V can be written as $\nabla \cdot V$ or div V as

$$\nabla \cdot V = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)(V_1 i + V_2 j + V_3 k)$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$$

Curl

if $v(x,y,z)$ is a vector field then the curl v or $\nabla \times v$ can be written as

$$\nabla \times v = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \times (v_1 i + v_2 j + v_3 k)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ v_1 & v_2 \end{vmatrix} k$$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) i - \left(\frac{\partial v_3}{\partial x} - \frac{\partial}{\partial z} v_1\right) j +$$

$$\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) k$$

properties

$$\nabla \cdot (\nabla \times A) = 0$$

$$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$$

$$\nabla \times (\phi A) = (\nabla \phi) \times A + \phi (\nabla \times A)$$

$$\nabla (A \cdot B) = (B \cdot \nabla) A + (A \cdot \nabla) B + B \times (\nabla \times A) + A \times (\nabla \times B).$$

Examples

1. if $\phi(x,y,z) = 3x^2y - y^3z^2$ find $\nabla\phi$ (grad. ϕ) at the point $(1,-2,-1)$

solu.

$$\nabla\phi = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)(3x^2y - y^3z^2)$$

$$= 6xy\,i + (3x^2 - 3y^2z^2)j - 2y^3z\,k.$$

$$= 6(1)(-2)i + [3(1)^2 - 3(-2)^2(-1)^2]j - 2(-2)^3(-1)k$$

$$= -12i - 9j - 16k.$$

2. Find $\nabla\phi$ if $\phi = \ln|r|$

solu.

$r = xi + yj + zk \qquad |r| = \sqrt{x^2+y^2+z^2}$

and $\phi = \ln|r| = \frac{1}{2}\ln(x^2+y^2+z^2)$

$$\nabla\phi = \frac{1}{2}\nabla\ln(x^2+y^2+z^2)$$

$$= \frac{1}{2}\left\{i\frac{\partial}{\partial x}\ln(x^2+y^2+z^2 + j\frac{\partial}{\partial y}\ln(x^2+y^2+z^2) + k\frac{\partial}{\partial z}\ln(x^2+y^2+z^2)\right\}$$

$$= \frac{1}{2}\left\{i\frac{2x}{x^2+y^2+z^2} + j\frac{2y}{x^2+y^2+z^2} + k\frac{2z}{x^2+y^2+z^2}\right\}$$

$$= \frac{xi + yj + zk}{x^2+y^2+z^2} = \frac{r}{r^2}$$

3. if $A = x^2 z\, i - 2y^3 z^2\, j + xy^2 z\, k$. Find $\nabla \cdot A$ (div A) at the point $(1,-1,1)$

Solu.

$$\nabla \cdot A = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot \left(x^2 z\, i - 2y^3 z^2\, j + xy^2 z\, k\right)$$

$$= \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(-2y^3 z^2) + \frac{\partial}{\partial z}(xy^2 z)$$

$$= 2xz - 6y^2 z^2 + xy^2 = 2(1)(1) - 6(-1)^2(1)^2 + (1)^2(-1)^2$$

$$= -3$$

H.W

4. if $A = xz^3\, i - 2x^2 yz\, j + 2yz^4\, k$ find $\nabla \times A$ at the point $(1,-1,1)$

Solu.

$$\nabla \times A = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \times \left(xz^3\, i - 2x^2 yz\, j + 2yz^4 k\right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2 yz & 2yz^4 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2x^2 yz)\right] i + \left[\frac{\partial}{\partial z}(xz^3) - \frac{\partial}{\partial x}(2yz^4)\right] j$$

$$+ \left[\frac{\partial}{\partial x}(-2x^2 yz) - \frac{\partial}{\partial y}(xz^3)\right] k$$

$$= (2x^4 + 2x^2 y)\, i + 3xz^2\, j - 4xyz\, k$$

$$= 3j + 4k.$$