Homogeneous Higher Order Differential Equation

It is homogeneous if $F(x) = 0$

Example

$$y''' - 6y'' + 11y' - 6y = 0$$

$$D^3 - 6D^2 + 11D - 6 = 0$$

Factors of 6 are: $(\pm 1, \quad \pm 2, \quad \pm 3, \quad \pm 6)$.

$$f(1) = (1)^3 - 6(1)^2 + 11(1) - 6 = 0 \quad \Rightarrow \quad r_1 = 1.$$

$$f'(D) = 3D^2 - 12D + 11$$

$$f'(1) = 3(1)^2 - 12(1) + 11 = 2 \neq 0 \quad \Rightarrow \quad r_1 = 1 \text{ is not a repeated root.}$$

$$\begin{array}{r|rrrr}
 & D^2 & -5D & +6 & \\
\hline
D \quad -1 & D^3 & -6D^2 & +11D & -6 \\
 & \mp D^3 & \pm D^2 & & \\
\hline
 & & -5D^2 & +11D & \\
 & & \pm 5D^2 & \mp 5D & \\
\hline
 & & & 6D & -6 \\
 & & & \mp 6D & \pm 6 \\
\hline
 & & & 0 & 0
\end{array}$$

$$D^2 - 5D + 6 = 0$$

$$(D-2)(D-3) = 0 \quad \Rightarrow \quad r_2 = 2 \text{ and } r_3 = 3$$

$$y_h = C_1 e^{r_1 x} + C_2 e^{r_2 x} + C_3 e^{r_3 x}$$

$$y_h = C_1 e^{x} + C_2 e^{2x} + C_3 e^{3x}$$

Example

$$y''' - 6y'' + 2y' + 36y = 0 \quad \Rightarrow \quad D^3 - 6D^2 + 2D + 36 = 0$$

Factors of 36 are: $(\pm 1, \quad \pm 2, \quad \pm 3, \quad . \quad . \quad .)$

$$f(1) = (1)^3 - 6(1)^2 + 2(1) + 36 = 33 \neq 0$$

$$f(-1) = (-1)^3 - 6(-1)^2 + 2(-1) + 36 = 27 \neq 0$$

$$f(2) = (2)^3 - 6(2)^2 + 2(2) + 36 = 24 \neq 0$$

$$f(-2) = (-2)^3 - 6(-2)^2 + 2(-2) + 36 = 0 \quad \Rightarrow \quad r_1 = -2$$

$$f'(D) = 3D^2 - 12D + 2$$

$$f'(-2) = 3(-2)^2 - 12(-2) + 2 = 38 \neq 0 \quad \Rightarrow \quad r_1 = -2 \text{ is not a repeated root.}$$

$$\begin{array}{r|rrrr}
 & D^2 & -8D & +18 & \\
\hline
D \quad +2 & D^3 & -6D^2 & +2D & +36 \\
 & \mp D^3 & \mp 2D^2 & & \\
\hline
 & & -8D^2 & +2D & \\
 & & \pm 8D^2 & \pm 16D & \\
\hline
 & & & 18D & +36 \\
 & & & \mp 18D & \mp 36 \\
\hline
 & & & 0 & 0
\end{array}$$

$$D^2 - 8D + 18 = 0$$

$$r_{2,3} = \frac{-B \mp \sqrt{B^2 - 4AC}}{2A} = \frac{8 \mp \sqrt{(8)^2 - 4(1)(18)}}{2(1)} = \frac{8 \mp \sqrt{64 - 72}}{2}$$

$$r_{2,3} = 4 \mp j\sqrt{2} \quad \Rightarrow \quad r_2 = 4 + j\sqrt{2} \quad \& \quad r_3 = 4 - j\sqrt{2}$$

$$\Rightarrow \quad \alpha = 4 \quad \& \quad \beta = \sqrt{2}$$

$$y_h = C_1 e^{r_1 x} + e^{\alpha x}\left(C_2 \cos(\beta x) + C_3 \sin(\beta x)\right)$$

$$y_h = C_1 e^{-2x} + e^{4x}\left(C_2 \cos\left(\sqrt{2}x\right) + C_3 \sin\left(\sqrt{2}x\right)\right)$$

$$v_2 = \int \frac{D_2}{D}\,dx = -\int \tan(x)dx = -\int \frac{\sin(x)}{\cos(x)}\,dx = \ln|\cos(x)|$$

$$v_3 = \int \frac{D_3}{D}\,dx = -\int dx = -x$$

$$y_p = v_1u_1 + v_2u_2 + v_3u_3$$

$$= \ln|\sec(x) + \tan(x)| + (\ln|\cos(x)|)\sin(x) - x\cos(x)$$

The general (complete) solution is

$$y = y_h + y_p$$

$$= C_1 + C_2\sin(x) + C_3\cos(x) + \ln|\sec(x) + \tan(x)| + (\ln|\cos(x)|)\sin(x) - x\cos(x)$$

Example

$$y''' - y' = 4x^3 + 6x^2$$

The homogeneous solution is found by solving $y''' - y' = 0$

$$D^3 - D = 0 \qquad \Rightarrow \quad D(D^2 - 1) = 0$$

$$D(D-1)(D+1) = 0 \;\Rightarrow\; r_1 = 0,\;\; r_2 = 1 \;\&\; r_3 = -1$$

$$y_h = C_1 + C_2e^x + C_3e^{-x}$$

To find the particular solution, let

$$y_p = x(Ax^3 + Bx^2 + Cx + D) = Ax^4 + Bx^3 + Cx^2 + Dx$$

$$y_p' = 4Ax^3 + 3Bx^2 + 2Cx + D \quad \Rightarrow \quad y_p'' = 12Ax^2 + 6Bx + 2C$$

$$y_p''' = 24Ax + 6B$$

$$(24Ax+6B)-(4Ax^3+3Bx^2+2Cx+D)=4x^3+6x^2$$

$$-4A=4 \quad \Rightarrow \quad A=-1$$

$$-3B=6 \quad \Rightarrow \quad B=-2$$

$$24A-2C=0 \quad \Rightarrow \quad 24(-1)-2C=0 \quad \Rightarrow \quad C=-12$$

$$6B-D=0 \quad \Rightarrow \quad 6(-2)-D=0 \quad \Rightarrow \quad D=-12$$

$$\Rightarrow \quad y_p=-x^4-2x^3-12x^2-12x$$

The general solution is

$$y=y_h+y_p=C_1+C_2e^x+C_3e^{-x}-x^4-2x^3-12x^2-12x$$

Example

$$y^{(4)}-8y''+16y=-18\sin(x)$$

The homogeneous solution is found using

$$y^{(4)}-8y''+16y=0 \quad \Rightarrow \quad D^4-8D^2+16=0$$

$$(D^2-4)^2=0 \quad \Rightarrow \quad r_{1,2}=\pm 2 \ \& \ r_{3,4}=\pm 2$$

$$y_h=C_1e^{2x}+C_2e^{-2x}+C_3xe^{2x}+C_4xe^{-2x}$$

To find the particular solution, let

$$y_p=A\cos(x)+B\sin(x) \quad \Rightarrow \quad y_p'=-A\sin(x)+B\cos(x)$$

$$y_p''=-A\cos(x)-B\sin(x) \quad \Rightarrow \quad y_p'''=A\sin(x)-B\cos(x)$$

$$y_p^{(4)}=A\cos(x)+B\sin(x)$$

$$(A\cos(x)+B\sin(x))-8(-A\cos(x)-B\sin(x))$$
$$+16(A\cos(x)+B\sin(x))=-18\sin(x)$$
$$(A+8A+16A)\cos(x)+(B+8B+16B)\sin(x)=-18\sin(x)$$
$$\Rightarrow\ 25A=0 \qquad \Rightarrow\ A=0$$
$$\Rightarrow\ 25B=-18 \quad \Rightarrow\ B=\frac{-18}{25}$$
$$\Rightarrow\ y_p=-\frac{18}{25}\sin(x)$$
$$y=y_h+y_p=C_1e^{2x}+C_2e^{-2x}+C_3xe^{2x}+C_4xe^{-2x}-\frac{18}{25}\sin(x)$$

Exercises

Find the solution of the following Differential Equations

1) $y''+y=3x^2$
2) $y''+2y'+y=x^2$
3) $y''+2y'+3y=27x$
4) $y''+y=-30\sin(4x)$
5) $y''+y=6\sin(x)$
6) $y''+4y'+3y=\sin(x)+2\cos(x)$
7) $y''+4y'+4y=18\cosh(x)$
8) $y''-2y'+2y=2e^x\cos(x)$
9) $y^{(4)}-5y''+4y=10\cos(x)$
10) $y''+y'-2y=3e^x$
11) $y''+y=x^2+x$
12) $y''-y=e^x$
13) $y''-2y'+y=e^x$
14) $y''+y'+y=x^4+4x^3+12x^2$
15) $y'''+2y''-y'-2y=1-4x^3$
16) $y''-2y'+2y=2e^x\cos(x)$