Multiple Integrals

Introduction « التكامل »

We know that the integral of a function of a single variable, say $y=f(x)$, over an interval $[a,b]$ is obtained as a Limit of approximating sums

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx$$

where $x_0=a<x_1<x_2<\cdots<x_n=b$ and c_i is any point from the interval $[x_{i-1}, x_i]$.

If $f(x)>0$, then $\int_a^b f(x)\,dx$ represent the area under the graph of $f(x)$ from $a=x$ to $x=b$

But in general the integral has many interpretation (such as distance, arc length, volume, surface area, moment of inertia, mass, hydrostatic, pressure, work, ...) depending on the nature and interpretation of $f(x)$.

Now, we shall define integrals of functions of two or more variables, which are called "multiple integrals" in the same way as integrals of functions of a single variable.

First, we shall define the integral of a function of two variables, say $z=f(x,y)$, whose graph in $\mathbb{R}^3$, over a rectangular region in the xy-plane and how such integral can be evaluated. Generalization of the definition to include bounded regions of more general nature will also discussed.

Double Integrals Over Rectangle Regions

Suppose that the function $z=f(x,y)$ is defined on a rectangular region R defined by:

$$R=\{(x,y): \quad a\le x\le b \quad , \quad c\le y\le d\}$$

If $f(x,y)$ is continuous throughout R, then the double integral of $f(x,y)$ over R is denoted by:

$$\iint_R f(x,y)\,dA = \lim_{\Delta A \to 0} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta A_i \quad ,$$

where $\Delta A = \Delta x \cdot \Delta y$.

Properties of The Double Integral

1. $\iint_R k\, f(x,y)\,dA = k \iint_R f(x,y)\,dA$, k is constant.

2. $\iint_R [f(x,y) \pm g(x,y)]\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$.

3. $\iint_R f(x,y)\,dA \geq 0$ if $f(x,y) \geq 0$ on R.

4. $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$ if $f(x,y) \geq g(x,y)$ on R

5. If $R = R_1 \sqcup R_2$ with $R_1 \cap R_2 = \phi$, then

$$\iint_R f(x,y)\,dA) = \iint_{R_1 \sqcup R_2} f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA .$$

<u>Note</u> If $f(x,y) > 0$, we may interpret $\iint_R f(x,y)\,dA$ as the volume of the solid enclosed by R (the planes $x=a$, $x=b$; $y=c$, $y=d$) and the surface $z=f(x,y)$. That is

$$\text{Volume} \equiv V = \iint_R f(x,y)\,dA.$$ This fact was found by Guido Fubini (1879–1943), he published his work in (1907).

<u>Fubini's Theorem</u> (First Form)

If $f(x,y)$ is continuous function on the rectangular region $R=\{(x,y) : a \le x \le b,\ c \le y \le d\}$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

<u>EX.(1):</u>

Let $f(x,y) = 1 - 6x^2y$ be a function defined on the region $R=\{(x,y) : 0 \le x \le 2,\ -1 \le y \le 1\}$. Caculate $\iint_R f(x,y)\,dA$.

<u>Solu.</u>

$$\iint_R f(x,y)\,dA = \int_{x=0}^{2} \int_{y=-1}^{1} (1-6x^2y)\,dy\,dx = \int_0^2 \left(y - 3x^2y^2\right)\Big|_{-1}^{1} dx$$

$$= \int_0^2 \left[(1-3x^2) - (-1-3x^2)\right] dx = 2\int_0^2 dx = 2x\Big|_0^2 = 2(2-0) = 4.$$

OR

$$\iint_R f(x,y)\,dA = \int_{y=-1}^{1}\int_{x=0}^{2} (1-6x^2y)\,dx\,dy = \int_{-1}^{1} (x-2x^3y)\Big|_0^2\,dy$$

$$= \int_{-1}^{1} (2-16y)\,dy = (2y-8y^2)\Big|_{-1}^{1} = (2-8)-(-2-8)$$

$$= -6+10 = 4.$$

EX. (2) Evaluate $\int_{\pi}^{2\pi}\int_{0}^{\pi} (\sin x+\cos y)\,dx\,dy$ and

$$\int_{0}^{\pi}\int_{\pi}^{2\pi} (\sin x+\cos y)\,dy\,dx.$$

Solu.

$$\int_{\pi}^{2\pi}\int_{0}^{\pi} (\sin x+\cos y)\,dx\,dy = \int_{\pi}^{2\pi} (-\cos x + x\cos y\Big|_0^{\pi}\,dy$$

$$= \int_{\pi}^{2\pi} [(-\cos(\pi)+\pi\cos y)-(-\cos 0+0)]\,dy = \int_{\pi}^{2\pi} (2+\pi\cos y)\,dy$$

$$= 2y+\pi\sin y\Big|_{\pi}^{2\pi} = (4\pi+0)-(2\pi+0) = 2\pi.$$

Now,

$$\int_{0}^{\pi}\int_{\pi}^{2\pi} (\sin x+\cos y)\,dy\,dx = \int_{0}^{\pi} (y\sin x+\sin y\Big|_{\pi}^{2\pi}\,dx$$

$$= \int_{0}^{\pi} [(2\pi\sin x+0)-(\pi\sin x+0)]\,dx = \int_{0}^{\pi} \pi\sin x\,dx = -\pi\cos x\Big|_0^{\pi}$$

$$= 2\pi.$$

Ex. (3)

Find the volume under the plane $z = 4 - x - y$ over the region $R = \{(x,y) : 0 \le x \le 2, \quad 0 \le y \le 1\}$ in xy-plane.

Solu :

$$\text{Volume} \equiv V = \iint_R f(x,y)\, dA = \int_{x=0}^{2} \int_{y=0}^{1} (4 - x - y)\, dy\, dx$$

$$= \int_0^2 \left[4y - xy - \tfrac{1}{2}y^2\right]_0^1 dx = \int_0^2 \left(4 - x - \tfrac{1}{2}\right) dx = \int_0^2 \left(\tfrac{9}{2} - x\right) dx$$

$$= \tfrac{7}{2}x - \tfrac{1}{2}x^2 \Big|_0^2 = (7 - 2) - (0 - 0) = 5.$$

OR

$$V = \int_{y=0}^{1} \int_{x=0}^{2} (4 - x - y)\, dx\, dy = \int_0^1 \left(4x - \tfrac{1}{2}x^2 - xy\right)\Big|_0^2 dy$$

$$= \int_0^1 (8 - 2 - 2y)\, dy = \int_0^1 (6 - 2y)\, dy = 6y - y^2 \Big|_0^1 = (6 - 1) - (0 - 0)$$

$$= 5.$$

Double Integrals For Bounded Nonrectangular Regions

Fubini's Theorem (Stronger Form)

Let $f(x,y)$ be a continuous function on a region R

1. If R is defined as $R=\{(x,y): a\leq x\leq b,\ f_1(x)\leq y\leq f_2(x)\}$ with $f_1(x)$ and $f_2(x)$ are continuous functions on $[a,b]$, then

$$\iint_R f(x,y)\,dA = \int_{x=a}^{b}\int_{y=f_1(x)}^{f_2(x)} f(x,y)\,dy\,dx . \qquad \text{Fig. (1)}$$

2. If R is defined as $R=\{(x,y): g_1(y)\leq x\leq g_2(y),\ c\leq y\leq d\}$ with $g_1(y)$ and $g_2(y)$ are continuous functions on $[c,d]$, then

$$\iint_R f(x,y)\,dA = \int_{y=c}^{d}\int_{x=g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy . \qquad \text{Fig(2)}$$

Fig(1)

Fig(2)

Note

Again, we may interpret $\iint_R f(x,y)\,dA$ as the volume of solid enclosed by the region R and the surface $z=f(x,y)$.

That is

$$\text{Volume} \equiv V = \iint_R f(x,y)\,dA .$$

EX.(4)

Find the integral of the function $f(x,y)=x^2y$ over the region $R=\{(x,y):\ 2y^2 \leq x \leq 2y,\ 0 \leq y \leq 1\}$. Sketch the region of integration.

$$\iint_R f(x,y)\,dA = \int_{y=0}^{1}\int_{x=2y^2}^{2y} x^2y\,dx\,dy = \int_0^1 \frac{1}{3}x^3y\Big|_{2y^2}^{2y} dy = \int_0^1 \frac{y}{3}(8y^3-8y^6)\,dy$$

$$= \frac{8}{3}\int_0^1 (y^4-y^7)\,dy = \frac{8}{3}\left(\frac{y^5}{5}-\frac{y^8}{8}\right)\Big|_0^1 = \frac{8}{3}\left(\frac{1}{5}-\frac{1}{8}\right)=\frac{1}{5}.$$

EX.(5) Evaluate

$$\int_{y=0}^{\ln 8}\int_{x=0}^{1-y} e^{x+y}\,dx\,dy = \int_{y=0}^{\ln 8} e^x\cdot e^y\Big|_0^{1-y} dy$$

$$= \int_0^{\ln 8} e^y(e^{1-y}-1)\,dy = \int_0^{\ln 8}(e-e^y)\,dy = ey-e^y\Big|_0^{\ln 8}$$

$$= e\ln 8 - 8 + 1 = 3e\ln 2 - 7 .$$

Determining The limits of Integration :

If we want to evaluate $\iint_R f(x,y)\,dA$ over the region R, we consider the following steps:

1. Imagine a vertical (horizontal) Line L Cutting the region R in the direction of increasing y (x).

2. Integrate from the y-value (x-value) where L inters R to the y-value (x-value) where L leaves R.

Then

$$\iint_R f(x,y)\,dA = \int_{x=a}^{b}\int_{y=f_1(x)}^{f_2(x)} f(x,y)\,dy\,dx ,$$

or

$$\iint_R f(x,y)\,dA = \int_{y=c}^{d}\int_{x=g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy .$$

EX(6) Calculate $\iint_R \sin x \, dA$ where R is the triangle in xy-plane bounded by the x-axis, the line $y=x$, and the Line $x=1$

Solu. (1)

$R=\{(x,y) : 0 \le x \le 1 , 0 \le y \le x\}$

$$\iint_R \sin x \, dx = \int_{x=0}^{1}\int_{y=0}^{x} \sin x \, dy\, dx = \int_0^1 \sin x y \Big|_0^x = \int_0^1 x \sin x \, dx$$

$$= -x\cos x + \sin x \Big|_0^1 = (-\cos(1)+\sin(1)) - (0+0) = \sin(1) - \cos(1)$$

Solu. (2)

$R=\{(x,y) : y \le x \le 1 , 0 \le y \le 1\}$

$$\iint_R \sin x \, dA = \int_{y=0}^{1}\int_{x=y}^{1} \sin x \, dx\, dy$$

$$= \int_{y=0}^{1} -\cos x \Big|_y^1 dy = \int_0^1 (-\cos(1)+\cos y)\, dy$$

$$= -y\cos(1) + \sin y \Big|_0^1 = -\cos(1)+\sin(1) = \sin(1) - \cos(1).$$

EX.(7) Calculate $\iint_R \frac{\sin x}{x}\, dA$ where R is the triangle in xy-plane bounded by the x-axis, the line $y=x$, and the line $x=1$

Solution

As shown in EX.(6)

If $R=\{(x,y) : y \le x \le 1 ,\ 0 \le y \le 1\}$, then

$$\iint_R \frac{\sin x}{x}\, dA = \int_{y=0}^{1}\int_{x=y}^{1} \frac{\sin x}{x}\, dx\, dy .$$

But $\int_{x=y}^{1} \frac{\sin x}{x}\, dx$ can not be solved analytically

So we try $R=\{(x,y) : 0 \le x \le 1 ,\ 0 \le y \le x\}$

$$\iint_R \frac{\sin x}{x}\, dA = \int_{x=0}^{1}\int_{y=0}^{x} \frac{\sin x}{x}\, dy\, dx = \int_0^1 \frac{\sin x}{x}\, y \Big|_0^x dx$$

$$= \int_0^1 \frac{\sin x}{x}(x-0)\, dx = \int_0^1 \sin x\, dx$$

$$= -\cos x \Big|_0^1 = -(\cos(1) - \cos(0))$$
$$= 1 - \cos(1) .$$

EX.(8) Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the line $y=x$ and $x=1$ and whose top lies in the plane $z=f(x,y)=3-x-y$.

Solu.

$R=\{(x,y) : 0 \le x \le 1 ,\ 0 \le y \le x\}$

$$\text{Volume} \equiv V = \iint_R f(x,y)\,dA = \int_{x=0}^{1}\int_{y=0}^{x} (3-x-y)\,dy\,dx$$

$$= \int_0^1 \left(3y - xy - \tfrac{1}{2}y^2\right)\Big|_0^x dx = \int_0^1 \left(3x - x^2 - \tfrac{1}{2}x^2\right)dx = \int_0^1 \left(3x - \tfrac{3}{2}x^2\right)dx$$

$$= \tfrac{3}{2}x^2 - \tfrac{1}{2}x^3\Big|_0^1 = \tfrac{3}{2} - \tfrac{1}{2} = 1 .$$

<u>OR</u> $R = \{(x,y) : y \le x \le 1 \;,\; 0 \le y \le 1\}$

$$\text{Volume} \equiv V = \iint_R f(x,y)\,dA = \int_{y=0}^{1}\int_{x=y}^{1} (3-x-y)\,dx\,dy = \int_0^1 \left(3x - \tfrac{1}{2}x^2 - yx\right)\Big|_y^1 dy$$

$$= \int_0^1 \left[\left(3 - \tfrac{1}{2} - y\right) - \left(3y - \tfrac{1}{2}y^2 - y^2\right)\right] dy = \int_0^1 \left(\tfrac{5}{2} - 4y + \tfrac{3}{2}y^2\right)dy = \tfrac{5}{2}y - 2y^2 + \tfrac{1}{2}y^3\Big|_0^1$$

$$= 1$$

<u>EX.(9)</u> Find the volume of the solid whose base is the region in xy-plane bounded by the parabola $y = 2 - x^2$ and the line $y = x$, while the top of the solid is bounded by the plane $z = 3 - x$

<u>Solu.</u>

$y = x \quad , \quad y = 2 - x^2$

$x = 2 - x^2$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0 \quad \Rightarrow x = 1, -2$

$$\text{Volume} \equiv V = \iint_R f(x,y)\,dA = \int_{x=-2}^{1}\int_{y=x}^{2-x^2} (3-x)\,dy\,dx$$

$$= \int_{-2}^{1} (3-x)y\Big|_x^{2-x^2} dx = \int_{-2}^{1} (3-x)(2-x^2-x)\,dx$$

$$= \int_{-2}^{1} \left(x^3 - 2x^2 - 5x + 6\right)dx = \tfrac{1}{4}x^4 - \tfrac{2}{3}x^3 - \tfrac{5}{2}x^2 + 6x\Big|_{-2}^{1}$$

$$= \left(\tfrac{1}{4} - \tfrac{2}{3} - \tfrac{5}{2} + 6\right) - \left(4 + \tfrac{16}{3} - 10 - 12\right)$$

$$= \tfrac{16}{3} + \tfrac{38}{3} = \tfrac{54}{3} .$$

Area

Defn. If $f(x,y)=1$, then the area of the region R in xy-plane is given by the double integral as

$$\text{Area} \equiv A = \iint_R dA = \iint_R dx\,dy = \iint_R dy\,dx .$$

EX.(1) Find the area of the region R in the 1st quadrant bounded by $y=x$ and $y=x^2$

Solu.

$y=x \quad , \quad y=x^2$

$x^2=x \Rightarrow x^2-x=0$

$x(x-1)=0 \Rightarrow x=0,1$

$$A=\iint_R dA = \int_{x=0}^{1}\int_{y=x^2}^{x} dy\,dx$$

where $R=\{(x,y): 0\le x\le 1 \ , \ x^2\le y\le x\}$

$$= \int_0^1 y\Big|_{x^2}^{x} dx = \int_0^1 (x-x^2)\,dx = \frac{1}{2}x^2-\frac{1}{3}x^3\Big|_0^1 = \frac{1}{2}-\frac{1}{3}=\frac{1}{6}$$

or

$$A=\iint_R dA = \int_{y=0}^{1}\int_{x=y}^{\sqrt{y}} dx\,dy \quad , \text{ where } R=\{(x,y): y\le x\le\sqrt{y}\ , 0\le y\le 1\}$$

$$= \int_0^1 x\Big|_y^{\sqrt{y}} dy = \int_0^1 (\sqrt{y}-y)\,dy = \frac{y^{3/2}}{3/2}-\frac{1}{2}y^2\Big|_0^1 = \frac{2}{3}-\frac{1}{2}=\frac{1}{6}$$

Ex(2) Find the area of the region R enclosed by the parabola $y=x^2$ and the line $y=x+2$.

Solu.

$y=x^2$, $y=x+2$

$x^2=x+2$

$x^2-x-2=0$

$(x-2)(x+1)=0 \Rightarrow x=-1,2$.

$$A=\iint_R dA=\int_{-1}^{2}\int_{y=x^2}^{x+2} dy\,dx \quad \text{where } R=\{(x,y):\ -1\le x\le 2\ ,\ x^2\le y\le x+2\}$$

$$=\int_{-1}^{2} y\Big|_{y=x^2}^{x+2} dx=\int_{-1}^{2}(x+2-x^2)\,dx=\frac{1}{2}x^2+2x-\frac{1}{3}x^3\Big|_{-1}^{2}$$

$$=\left(2+4-\frac{8}{3}\right)-\left(\frac{1}{2}-2+\frac{1}{3}\right)=\frac{9}{2}.$$

or

$$A=\iint_R dA=\iint_{R_1} dA+\iint_{R_2} dA$$

$$=\int_{y=0}^{1}\int_{x=-\sqrt{y}}^{\sqrt{y}} dx\,dy+\int_{y=1}^{4}\int_{x=y-2}^{\sqrt{y}} dx\,dy$$

where $R_1=\{(x,y):-\sqrt{y}\le x\le\sqrt{y}\ ,0\le y\le 1\}$

$R_2=\{(x,y):\ y-2\le x\le\sqrt{y}\ ,1\le y\le 4\}$

$$= \int_0^1 x \Big|_{-\sqrt{y}}^{\sqrt{y}} dy + \int_1^4 x \Big|_{y-2}^{\sqrt{y}} dy = \int_0^1 (\sqrt{y}+\sqrt{y})\,dy + \int_1^4 (-y+2+\sqrt{y})\,dy$$

$$\therefore \quad A = 2\int_0^1 \sqrt{y}\,dy + \int_1^4 (\sqrt{y} - y + 2)\,dy = 2\,\frac{y^{3/2}}{3/2}\Big|_0^1 + \left(\frac{y^{3/2}}{3/2} - \frac{1}{2}y^2 + 2y\right)\Big|_1^4$$

$$= \frac{4}{3}(1-0) + \left(\frac{16}{3} - 8 + 8\right) - \left(\frac{2}{3} - \frac{1}{2} + 2\right) = \frac{4}{3} + \frac{16}{3} - \frac{13}{6} = \frac{20}{3} - \frac{13}{6}$$

$$= \frac{40-13}{6} = \frac{27}{6} = \frac{9}{2}.$$

Change of Variables In Double Integrals

Some Cartesian double integrals can not be integrated unless it is transformed to another region of integration.

Consider the double integral $\iint_R f(x,y)\,dx\,dy$. Transformation can be made by introducing a new variables of integration, say u and v, by setting $x = g_1(u,v)$ and $y = g_2(u,v)$, where g_1 and g_2 are continuous functions having 1st partial derivatives defined on some region R^* in uv-plane, then:

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f[g_1(u,v), g_2(u,v)]\,|J|\,dv\,du.$$

where $$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

is the determinant of partial derivatives and it is called the Jacobian of transformation.

Ex.(1) Use the transformation $x = u \cdot v$ and $y = v$ and evaluate

$$\iint_R f(x,y)\,dx\,dy \quad , \text{ where } R = \{(x,y) : 0 \le x < \infty, 0 \le y < \infty\}$$

and $f(x,y) = e^{-\frac{1}{2}(x^2+y^2)}$, $(x,y) \in R$.

Solu.

$x = u \cdot v$, $y = v$ $\Rightarrow$ $u = \frac{x}{y}$, $v = y$, then the region of integration in uv-plane is

$$R^* = \{(u,v) : 0 \le u < \infty , 0 \le v < \infty\}$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

$$\Rightarrow |J| = |v| = v$$

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f(u \cdot v, v)\,|J|\,dv\,du = \int_{x=0}^{\infty}\int_{y=0}^{\infty} e^{-\frac{1}{2}(x^2+y^2)}\,dx\,dy$$

$$= \int_{u=0}^{\infty}\int_{v=0}^{\infty} e^{-\frac{1}{2}(u^2v^2+v^2)} \cdot v \, dv\,du$$

$$= \int_{u=0}^{\infty}\int_{v=0}^{\infty} e^{-\frac{1}{2}(u^2+1)v^2} \cdot v \, dv\,du$$

$$= \int_{u=0}^{\infty} -\frac{1}{u^2+1}\, e^{-\frac{1}{2}(u^2+1)v^2}\Big|_0^{\infty} du = \int_{u=0}^{\infty} -\frac{1}{u^2+1}\left(e^{-\infty} - e^{0}\right) du = \int_0^{\infty} \frac{du}{u^2+1}$$

$$= \tan^{-1} u \Big|_0^{\infty} = \tan^{-1}(\infty) - \tan^{-1}(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

<u>EX.(2)</u> Use the transformation $x = (-2\ln u)^{\frac{1}{2}} \cos(2\pi v)$, $y = (-2\ln u)^{\frac{1}{2}} \sin(2\pi v)$ and evaluate $\iint_R f(x,y)\,dx\,dy$ where

$R = \{(x,y) : 0 < x < \infty \ , \ 0 < y < \infty\}$ and

$$f(x,y) = e^{-\frac{1}{2}(x^2+y^2)}, \quad (x,y) \in R$$

<u>Solu.</u>

$x = (-2\ln u)^{1/2} \cos(2\pi v) \quad , \quad y = (-2\ln u)^{1/2} \sin(2\pi v)$

$$x^2 + y^2 = (-2\ln u)\cos^2(2\pi v) + (-2\ln u)\sin^2(2\pi v)$$
$$= (-2\ln u)\left[\cos^2(2\pi v) + \sin^2(2\pi v)\right] = -2\ln u$$

$$\Rightarrow \ln u = -\frac{1}{2}(x^2+y^2) \Rightarrow u = e^{-\frac{1}{2}(x^2+y^2)}$$

$$\frac{y}{x} = \tan(2\pi v) \Rightarrow 2\pi v = \tan^{-1}\left(\frac{y}{x}\right) \Rightarrow v = \frac{1}{2\pi}\tan^{-1}\left(\frac{y}{x}\right)$$

So the region of integration in uv-plane is

$R^* = \{(u,v) : 0 < u < 1 \ , \ 0 < v < \frac{1}{4}\}$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2}(-2\ln u)^{-\frac{1}{2}} \cos(2\pi v)\left(-\frac{2}{u}\right) & -2\pi(-2\ln u)^{1/2} \sin(2\pi v) \\ -\frac{1}{2}(-2\ln u)^{-\frac{1}{2}} \sin(2\pi v)\left(-\frac{2}{u}\right) & 2\pi(-2\ln u)^{1/2} \cos(2\pi v) \end{vmatrix}$$

$$= -\frac{2\pi}{u}\cos^2(2\pi v) - \frac{2\pi}{u}\sin^2(2\pi v) = -\frac{2\pi}{u} \Rightarrow |J| = \frac{2\pi}{u}$$

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f\left[(-2\ln u)^{1/2}\cos(2\pi v),\ (-2\ln u)^{1/2}\sin(2\pi v)\right]|J|\,du\,dv$$

$$\Rightarrow \int_{x=0}^{\infty}\int_{y=0}^{\infty} e^{-\frac{1}{2}(x^2+y^2)}\,dx\,dy = \int_{u=0}^{1}\int_{v=0}^{\frac{1}{4}} u\,\frac{2\pi}{u}\,dv\,du = 2\pi\int_{u=0}^{1}\int_{v=0}^{\frac{1}{4}} dv\,du$$

$$= 2\pi\int_0^1 v\Big|_0^{\frac{1}{4}} du = \frac{2\pi}{4}\int_0^1 du = \frac{\pi}{2}\,u\Big|_0^1 = \frac{\pi}{2}(1-0) = \frac{\pi}{2}$$

<u>EX.(3)</u> Use the transformation $x = u + \frac{v}{2}$, $y = v$ and evaluate $\iint_R f(x,y)\,dx\,dy$ where $R = \{(x,y) : \frac{y}{2} \le x \le \frac{y}{2}+2,\ 0 \le y \le 2\}$

and

$$f(x,y) = y^3(2x-y)\,e^{(2x-y)^2}, \quad (x,y)\in R.$$

<u>Solu.</u>

$x = u + \frac{v}{2}$, $y = v$ $\Rightarrow u = x - \frac{y}{2} = \frac{1}{2}(2x-y)$, $v = y$. Then the region of integration in uv-plane is

$R^* = \{(u,v) : 0 \le u \le 2,\ 0 \le v \le 2\}$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1 \Rightarrow |J| = 1$$

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f\left(u+\frac{v}{2}, v\right)|J|\,du\,dv \Rightarrow$$

$$\int_{y=0}^{2}\int_{x=\frac{y}{2}}^{\frac{y}{2}+2} y^3(2x-y)\,e^{(2x-y)^2}\,dx\,dy = \int_{u=0}^{2}\int_{v=0}^{2} v^3 \cdot 2u\,e^{4u^2}\,dv\,du.$$

$$= \int_{u=0}^{2} 2u\,e^{4u^2}\,\frac{v^4}{4}\Big|_0^2\,du = \int_0^2 2u\,e^{4u^2}(4-0)\,du$$

$$= \int_0^2 e^{4u^2}\,8u\,du = e^{4u^2}\Big|_0^2 = e^{16} - e^0 = e^{16} - 1.$$

Evaluation of Double By Changing To Polar Coordinates

We shall introduce change of variables for double integrals which transform from the Cartesian coordinates to polar coordinates by setting:

$$x = r\cos\theta \quad , \quad y = r\sin\theta .$$

Where r and θ are the new integration variables that, introduced, with Jacobian determinant of transformation

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r \quad \text{and } |J| = r.$$

Therefore

$$\iint_{R^*} f(x,y)\,dx\,dy = \iint_{R^*} f(r\cos\theta, r\sin\theta)\,|J|\,dr\,d\theta$$

$$= \iint_{R^*} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

where R^* is the region of integration in polar coordinates

<u>Note</u>

$r^2 = x^2 + y^2$

$\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$.

$\sin\theta = \frac{y}{r}$

$\Rightarrow y = r\sin\theta$

$\cos\theta = \frac{x}{r}$

$\Rightarrow x = r\cos\theta$.

<u>EX.(4)</u> Change to polar and evaluate $\iint_R f(x,y)\,dx\,dy$

where $R = \{(x,y) : -1 \le x \le 1 \;,\; -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\}$

and $f(x,y) = 1$

<u>Solu.</u>

We may solve as :

$$\iint_R f(x,y)\,dx\,dy = \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx = \int_{-1}^{1} y \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} \left(\sqrt{1-x^2} + \sqrt{1-x^2}\right) dx = \int_{-1}^{1} 2\sqrt{1-x^2}\,dx$$

$x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\,\cos\theta\, d\theta = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta = \cancel{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{\cancel{2}}\, d\theta$$

$$= \theta + \frac{1}{2}\sin 2\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \left(\frac{\pi}{2} - 0\right) - \left(-\frac{\pi}{2} + 0\right) = \pi .$$

Changing to polar by setting $x = r\cos\theta$, $y = r\sin\theta$.

$$\iint_R f(x,y)\,dx\,dy = \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx = \iint_{R^*} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

where $R^* = \{(r,\theta) : 0 \le r \le 1 , 0 \le \theta \le 2\pi\}$

So

$$\iint_R f(x,y)\,dx\,dy = \int_{r=0}^{1}\int_{\theta=0}^{2\pi} r\,d\theta\,dr = \int_0^1 r\theta \Big|_0^{2\pi} dr = 2\pi\int_0^1 r\,dr$$

$$= 2\pi\, \frac{1}{2} r^2 \Big|_0^1 = 2\pi\, \frac{1}{2}(1-0) = \pi .$$

<u>EX.(5)</u> Change to polar and evaluate $\iint_R f(x,y)\,dx\,dy$ where

$R = \{(x,y) : 0 < x < \infty , 0 < y < \infty\}$ and

$f(x,y) = e^{-\frac{1}{2}(x^2+y^2)}$, $(x,y) \in R$.

Solu.

Changing to polar by setting $x = r\cos\theta$, $y = r\sin\theta$.

The region of integration in polar coordinates is

$$R^* = \{(r,\theta) : 0 \le r < \infty \ , \ 0 \le \theta \le \tfrac{\pi}{2}\}$$

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

So
$$\int_{x=0}^{\infty}\int_{y=0}^{\infty} e^{-\frac{1}{2}(x^2+y^2)}\,dy\,dx = \int_{r=0}^{\infty}\int_{\theta=0}^{\frac{\pi}{2}} e^{-\frac{1}{2}r^2}\, r\,d\theta\,dr$$

$$= \int_{r=0}^{\infty} e^{-\frac{1}{2}r^2}\, r\,\theta \Big|_0^{\frac{\pi}{2}} dr = \frac{\pi}{2}\int_0^{\infty} e^{-\frac{1}{2}r^2}\, r\,dr = -\frac{\pi}{2}\, e^{-\frac{1}{2}r^2}\Big|_0^{\infty}$$

$$= -\frac{\pi}{2}\left(e^{-\infty} - 1\right) = \frac{\pi}{2}.$$

Ex.(6) Integrate the function $f(x,y) = \dfrac{1}{1-x^2-y^2}$ over the disk $x^2+y^2 \le \frac{3}{4}$.

Solu.

We have $R = \{(x,y) : x^2+y^2 \le \frac{3}{4}\}$

OR $R = \{(x,y) : -\frac{\sqrt{3}}{2} \le x \le \frac{\sqrt{3}}{2} \ , \ -\sqrt{\frac{3}{4}-x^2} \le y \le \sqrt{\frac{3}{4}-x^2}\}$

So
$$\iint_R f(x,y)\,dx\,dy = \int_{x=-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \int_{y=-\sqrt{\frac{3}{4}-x^2}}^{\sqrt{\frac{3}{4}-x^2}} \frac{dx\,dy}{1-x^2-y^2}.$$

Chang to polar by setting $x = r\cos\theta$, $y = r\sin\theta$.

Under polar transformation we have the region of integration

$$R^* = \{(r,\theta) : 0 \leq r \leq \tfrac{\sqrt{3}}{2}, \quad 0 \leq \theta \leq 2\pi\}$$

So
$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

$$= \int_{r=0}^{\frac{\sqrt{3}}{2}} \int_{\theta=0}^{2\pi} \frac{r\,d\theta\,dr}{1-r^2} = \int_{r=0}^{\frac{\sqrt{3}}{2}} \frac{r}{1-r^2}\,\theta \Big|_0^{2\pi} dr$$

$$= 2\pi \int_{r=0}^{\sqrt{3}/2} \frac{r\,dr}{1-r^2} = 2\pi\left(-\frac{1}{2}\right) \ln(1-r^2) \Big|_0^{\sqrt{3}/2}$$

$$= -\pi\left[\ln\left(1-\frac{3}{4}\right) - \ln(1)\right] = -\pi\left[\ln\left(\frac{1}{4}\right) - 0\right]$$

$$= \pi \ln 4 = \pi \ln 2^2 = 2\pi \ln 2.$$

<u>EX. (7)</u>: Integrate the function $f(x,y) = \dfrac{\ln(x^2+y^2)}{(x^2+y^2)}$ over the region between circles $x^2+y^2=r$ and $x^2+y^2=e^2$.

Solu.

$\iint_R f(x,y)\,dx\,dy$ where

$f(x,y)=\dfrac{\ln(x^2+y^2)}{x^2+y^2}$ and

$R=R_1 \sqcup R_2$ where

$R_1=\{(x,y): 1<x<e,\ +\sqrt{1-x^2}\le y\le +\sqrt{e^2-x^2}\}$

$R_2=\{(x,y): 1<x<e,\ -\sqrt{1-x^2}\le y\le -\sqrt{e^2-x^2}\}$.

So

$$\iint_R f(x,y)\,dx\,dy=\iint_{R_1} f(x,y)\,dx\,dy+\iint_{R_2} f(x,y)\,dx\,dy$$

Change to polar by setting $x=r\cos\theta$, $y=r\sin\theta$.

$x^2+y^2=1 \Rightarrow r^2=1 \Rightarrow r=1$

$x^2+y^2=e^2 \Rightarrow r^2=e^2 \Rightarrow r=e$

So the region of integration in polar is

$R^*=\{(r,\theta): 1\le r\le e,\ 0\le\theta\le 2\pi\}$

$$\therefore \iint_R f(x,y)\,dx\,dy=\iint_{R^*} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

$$\Rightarrow \iint_R f(x,y)\,dx\,dy=\int_{r=1}^{e}\int_{\theta=0}^{2\pi}\frac{2\ln r}{r}\,d\theta\,dr=4\pi\int_1^e\frac{\ln r}{r}\,dr$$

$$=\cancel{4}\pi\,\frac{(\ln r)^2}{\cancel{2}}\Big|_1^e=2\pi\left[(\ln e)^2-(\ln(1))^2\right]=2\pi(1-0)$$

$$=2\pi.$$

EX.(8) Find the volume of the solid in the 1st octant under the graph of $z=f(x,y)=3-\sqrt{x^2+y^2}$ and inside the circle $(x-1)^2+y^2=1$

Solu.

The circle $(x-1)^2+y^2=1$ has center at $(1,0)$ and radius $r=1$.

$(x-1)^2+y^2=1 \Rightarrow x^2-2x+1+y^2=1$

$\Rightarrow y^2=2x-x^2 \quad \Rightarrow y=\sqrt{2x-x^2}$

So $R=\{(x,y) : 0\le x\le 2 \ , \ 0\le y\le \sqrt{2x-x^2}\}$

$$\text{Volume} \equiv V=\iint_R f(x,y)\,dx\,dy=\int_{x=0}^{2}\int_{y=0}^{\sqrt{2x-x^2}}(3-\sqrt{x^2+y^2})\,dy\,dx$$

Change to polar by setting $x=r\cos\theta$, $y=r\sin\theta$

we have $0\le\theta\le\frac{\pi}{2}$, we requaired

$? \le r \le ?$

Since $(x-1)^2+y^2=1 \Rightarrow x^2-2x+1+y^2=1$

$\Rightarrow x^2+y^2-2x=0 \Rightarrow r^2-2r\cos\theta=0$

$r(r-2\cos\theta)=0 \Rightarrow r=0$ or $r-2\cos\theta=0$

$\Rightarrow r=2\cos\theta$ So $0\le r\le 2\cos\theta$

Therefore the region of integration in polar is

$R^*=\{(r,\theta) : 0\le r\le 2\cos\theta \ , \ 0\le\theta\le\frac{\pi}{2}\}$

$$\therefore V=\iint_{R^*} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta .$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{2\cos\theta} (3-r)\, r\, dr\, d\theta = \int_{\theta=0}^{\frac{\pi}{2}} \left(\frac{3}{2} r^2 - \frac{1}{3} r^3 \right|_0^{2\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{3}{2} \cdot 4\cos^2\theta - \frac{1}{3} \cdot 8\cos^3\theta \right) d\theta = \int_0^{\frac{\pi}{2}} \left(6\cos^2\theta - \frac{8}{3}\cos^3\theta \right) d\theta$$

$$= 6 \int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\, d\theta - \frac{8}{3} \int_0^{\pi/2} \cos\theta (1-\sin^2\theta)\, d\theta$$

$$= 3 \left(\theta + \frac{1}{2}\sin 2\theta \right|_0^{\frac{\pi}{2}} - \frac{8}{3} \int_0^{\frac{\pi}{2}} (1-\sin^2\theta)\cos\theta\, d\theta$$

$$= \frac{3\pi}{2} - \frac{8}{3} \left(\sin\theta - \frac{\sin^3\theta}{3} \right|_0^{\frac{\pi}{2}} = \frac{3}{2}\pi - \frac{8}{3}\left(1 - \frac{1}{3}\right).$$

$$= \frac{3}{2}\pi - \frac{16}{9}$$

Exercises

Evaluate the following integrals:

(1) $\int_0^3 \int_0^2 (4-y^2)\,dy\,dx$ (2) $\int_{-1}^{0} \int_{-1}^{1} (x+y+1)\,dx\,dy$

(3) $\int_0^3 \int_{-2}^{0} (x^2y-2xy)\,dy\,dx$ (4) $\int_0^{\pi} \int_0^{x} x\sin y\,dy\,dx$

(5) $\int_0^{\pi} \int_0^{\sin x} y\,dy\,dx$ (6) $\int_1^2 \int_y^{y^2} dx\,dy$ (7) $\int_0^1 \int_0^{\frac{1}{y}} y\,e^{xy}\,dx\,dy$

(8) $\int_0^1 \int_0^{x^3} e^{\frac{y}{x}}\,dy\,dx$ (9) $\int_0^1 \int_{\sqrt{y}}^{2-\sqrt{y}} xy\,dx\,dy$

(10) $V = \int_0^1 \int_0^y (x^2+y^2)\,dx\,dy + \int_1^2 \int_1^{2-y} (x^2+y^2)\,dx\,dy$.

In the following, integrate the function $f(x,y)$ over the given region

(11) $f(x,y) = \frac{x}{y}$ over the region in the 1st quadrant bounded by the lines $y=x$, $y=2x$, $x=1$, $x=2$.

(12) $f(x,y) = x^2+y^2$ over the triangular region whose vertices are $(0,0)$, $(1,0)$ and $(0,1)$.

(13) $f(x,y) = y-\sqrt{x}$ over the triangular region cut from the 1st quadrant by the line $x+y=1$.

(14) $f(x,y) = x^2 + 3xy$ over the region $R = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1\}$. (28)

(15) $f(x,y) = (xy)^{-1}$ over the region $R = \{(x,y) : 1 \le x \le 2,\ 1 \le y \le 2\}$.

(16) $f(x,y) = y\cos(xy)$ over the region $R = \{(x,y) : 0 \le x \le \pi,\ 0 \le y \le 1\}$

In the following, <u>Skeck</u> the region over which the integration takes place and write an equivalent integral with order of integration <u>reversed</u>. Evaluate <u>both</u> integrations:

(17) $\displaystyle\int_0^2 \int_1^{e^x} dy\,dx$ (18) $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 dx\,dy$ (19) $\displaystyle\int_0^{\sqrt{2}} \int_{-\sqrt{1-2y^2}}^{\sqrt{1-2y^2}} y\,dx\,dy$.

(20) $\displaystyle\int_{-2}^1 \int_{x^2+4x}^{3x+2} dy\,dx$.

Evaluate the following integrals by integrating the equivalent integral obtained by <u>reversing</u> the order of integration

(21) $\displaystyle\int_0^1 \int_{2y}^2 \cos(x^2)\,dx\,dy$ (22) $\displaystyle\int_0^1 \int_y^1 x^2 e^{xy}\,dx\,dy$

(23) $\displaystyle\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y}\,dy\,dx$ (24) $\displaystyle\int_0^2 \int_x^2 y^2 \sin(xy)\,dy\,dx$

(25) $\displaystyle\int_0^8 \int_{x^{1/3}}^2 \frac{dy\,dx}{y^4+1}$ (26) $\displaystyle\int_0^2 \int_0^{4-x^2} \frac{x\,e^{2y}}{4-y}\,dx\,dy$.

(27) $\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\,dx\,dy$.

(28) Find the volume in the 1st octant bounded by coordinate plane, the cylinder $x^2+y^2=4$ and the plane $z=f(x,y)=3-y$.

(29) Find the volume of the solid in the 1st octant bounded by the coordinates planes, the plane $x=3$, and the parabolic cylinder $z=f(x,y)=4-y^2$.

(30) Evaluate $\int_0^2 [\tan^{-1}(\pi x)-\tan^{-1}x]\,dx$. Use double integral region.

Exercises

Find the area of the region bounded by the given curves and lines by means of double integration. Sketch

(1) The x-axis, $y=e^x$, $x=0$, and $x=1$.

(2) The y-axis, $y=2x$, and $y=4$.

(3) The curves $y^2=-x$, $y=x+2$.

(4) The curves $x=y^2$, $x=2y-y^2$.

(5) The curves $x=y-y^2$, $x+y=0$

(6) Above by $y=x^2$, below by $y=-1$, on the left by $x=-2$ and on the right by $y=2x-1$.