Matrices And Determinants

المصفوفات والمحددات

<u>Defn.</u> A rectangular array of real (or complex) numbers of the form :

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \quad \cdots (1)$$

is called a matrix. (المصفوفة)

- The numbers $a_{11}, a_{12}, \ldots, a_{mn}$ are called the elements of the matrix.
- The horizontal Lines are called rows (صفوف)
- The vertical Lines are called Columns (اعمدة)
- A matrix with m rows and n column is called $m \times n$ matrix (read m by n matrix).
- Matrices will be denoted by capital bold-faced Letters $A, B, \ldots$ or by $[a_{jk}], [b_{jk}], \ldots$
- A matrix $[a_1 \; a_2 \; \cdots \; a_n]$ having only one row is called a Row Matrix or row vector.
- A vector having only one column is called a column matrix or column vector. $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

- The elements of a matrix may be real or complex numbers. If all the elements of a matrix are real the matrix is called a Real Matrix (مصفوفة حقيقية).

- A matrix having the same number of rows and columns is called a square matrix (مصفوفة مربعة), and the number of rows is called its order. A square matrix of order n is called an $n \times n$ matrix (read n by n matrix).

The Types of Matrices انواع المصفوفات

① Square Matrix: المصفوفة المربعة

number of rows = number of columns

EX.

$$\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} -1 & 3 & 2 \\ 0 & 7 & 1 \\ 2 & 4 & 5 \end{bmatrix}$$

عدد الصفوف = عدد الاعمدة = 2

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② Diagonal Matrix المصفوفة القطرية

It is square matrix, all the elements are zero except the elements of the main diagonal are not zero.

EX.

$$\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

③ Triangular Matrix المصفوفة المثلثية

It is a square matrix whose elements upper or lower

EX.

$$\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}, \quad \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 5 & 0 \\ 4 & 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 7 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

4 Identity Matrix المصفوفة الاحادية (مصفوفة الوحدة)

Its a diagonal matrix whose elements on the main diagonal are all 1 and denoted by I_n

$$I_1 = [1] \quad , \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5 Zero Matrix المصفوفة الصفرية

All the elements are Zero

$$[0] \, , \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} , \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Mathematical Operations of Matrices

العمليات الرياضية للمصفوفات

Let : $A = [a_{jk}]_{m \times n}$ and $B = [b_{jk}]_{m \times n}$ ---(2)

1 Equality :

Two matrices A and B are equal if and only if A and B have the same number of rows and the same number of columns and corresponding elements are equal. That is $a_{jk} = b_{jk}$ for all occuring j and k.

2 Addition of Matrices :

Is defined only for matrices having the same number of rows and columns and is defined as follows :

The sum of two $m \times n$ matrices $A = [a_{jk}]$ and $B = [b_{jk}]$ is the $m \times n$ matrix $C = [c_{jk}]$ with elements :

$$c_{jk} = a_{jk} + b_{jk} \qquad j = 1, 2, \dots, m \qquad k = 1, 2, \dots, n \qquad \text{--- (3)}$$

and written $C = A + B$.

EX.(1) Addition of Matrices:

$$A = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

then

$$A + B = \begin{bmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{bmatrix}.$$

Properties of Matrix addition:

① $A + B = B + A$

② $A + 0 = A$ --- ④

③ $A + (-A) = 0$

where $-A = [-a_{jk}]$ is the $m \times n$ matrix obtained by multiplying every element of A by (-1) and is called the negative of A.

Multiplication of Matrices by Scalars (numbers).

The product of an $m \times n$ matrix A by a scalar q is denoted by qA or Aq and is the $m \times n$ matrix obtained by multiplying every element of A by q.

$$qA = Aq = \begin{bmatrix} qa_{11} & qa_{12} & \cdots & qa_{1n} \\ qa_{21} & qa_{22} & \cdots & qa_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ qa_{m1} & qa_{m2} & \cdots & qa_{mn} \end{bmatrix}$$

Ex. If $A = \begin{bmatrix} 2.7 & -1.8 \\ 0.9 & 3.6 \end{bmatrix}$ Calculate $2A$ and $\frac{10}{9}A$

Solu.

$$2A = \begin{bmatrix} 5.4 & -3.6 \\ 1.8 & 7.2 \end{bmatrix} \text{ and } \frac{10}{9}A = \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}$$

Properties of Matrix Multiplication by Scalars

$$\left.\begin{array}{ll} ① & q(A+B) = qA + qB \\ ② & (c+k)\cdot A = cA + kA \\ ③ & c(kA) = ck(A) \\ ④ & 1\cdot A = A \end{array}\right\} \quad \cdots (5)$$

where q, c, and k are scalars

Transpose of A Matrix : مدور المصفوفة

The matrix of order $n \times m$ obtained by interchanging the rows and columns of an $m \times n$ matrix is called the transpose of A and is denoted A^T.

i.e.

The transpose of $A = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

is $A^T = [a_{kj}] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix} \quad \cdots (6)$

For example:

The transpose of $A = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$ is $A^T = \begin{bmatrix} 3 & 6 \\ 4 & 7 \\ 5 & 8 \end{bmatrix}$

- A real square matrix $A = [a_{jk}]$ is said to be symmetric if it is equal to its transpose, $A^T = A$, that is $a_{kj} = a_{jk}$.

<u>For example</u>

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

then $A^T = A \Rightarrow A$ is symmetric matrix.

- A real square matrix $A = [a_{jk}]$ is said to be skew-symmetric if $A^T = -A$, that is

$$a_{kj} = -a_{jk}.$$

<u>For example</u>

$$A = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -5 \\ -1 & 5 & 0 \end{bmatrix}, \quad A^T = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 5 \\ 1 & -5 & 0 \end{bmatrix}$$

$$\therefore \quad A^T = -A$$

- A real square matrix A may be written as the sum of a symmetric matrix R and a skew-symmetric matrix S, where

$$\left.\begin{aligned} R &= \tfrac{1}{2}(A + A^T) \\ S &= \tfrac{1}{2}(A - A^T) \\ R + S &= A \end{aligned}\right\} \quad \text{--- (7)}$$

<u>Ex.</u> The matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}$ is symmetric

and $A = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$ is skew-symmetric

while The matrix $A = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$ is neither symmetric nor skew-symmetric.

Matrix Multiplication: ضرب المصفوفات

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$,

then the product of AB is the matrix C

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

where $c_{11} = a_{11} b_{11} + a_{12} b_{21}$, $c_{12} = a_{11} b_{12} + a_{12} b_{22}$

$c_{21} = a_{21} b_{11} + a_{22} b_{21}$, $c_{22} = a_{21} b_{12} + a_{22} b_{22}$

or briefly $c_{jk} = a_{j1} b_{1k} + a_{j2} b_{2k} = \sum_{i=1}^{2} a_{ji} b_{ik}$ --- (8)

EX. Evaluate the following product:

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 4 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 10+12+3 & 6+0+1 \\ -5+8+12 & -3+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 7 \\ 15 & 1 \end{bmatrix}$$

EX

$$\begin{bmatrix} 3 & 4 & 2 \\ 6 & 0 & -1 \\ -5 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3+12+4 \\ 6+0-2 \\ -5-6+2 \end{bmatrix} = \begin{bmatrix} 19 \\ 4 \\ -9 \end{bmatrix}$$

Ex.

$$\begin{bmatrix} 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 19 \end{bmatrix}$$

EX

$$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 10 & 12 \\ 12 & 15 & 18 \\ -4 & -5 & -6 \end{bmatrix}$$

Properties of Matrix Multiplication:

Assuming that A, B, C are conformable for the indicated sums and products. We have

$A(B+C) = AB + AC$ first distributive Law.

$(A+B)C = AC + BC$ second " ".

$A(BC) = (AB)C$ associative Law.

However,

$AB \neq BA$, generally

$AB = 0$ does not necessarily imply $A=0$ or $B=0$

$AB = AC$ " " " " $B = C$

Example:

Let $A = \begin{bmatrix} 4 & 1 & 2 & 1 \\ -3 & -1 & 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 4 & 1 \\ 1 & 1 & 6 & 0 \end{bmatrix}$

$C = \begin{bmatrix} x-1 & 3y \\ \frac{z}{2} & w-1 \\ q & p+1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 6 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$

Find

① $2A$, $-3B$, $\frac{2}{3}B$

② $A+B$, $A-B$, $2A-3B$

③ A^T, B^T, A^T+B^T, $2A^T$, $3B^T$

if $C=D$ find the values of x, y, z, w, q and p.

Solu:

① $2A = \begin{bmatrix} 8 & 2 & 4 & 2 \\ -6 & -2 & 0 & 6 \end{bmatrix}$, $-3B = \begin{bmatrix} -9 & 3 & -12 & -3 \\ -3 & -3 & -18 & 0 \end{bmatrix}$

$\frac{2}{3}B = \begin{bmatrix} 2 & -\frac{2}{3} & \frac{8}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & 4 & 0 \end{bmatrix}$.

2 $A+B = \begin{bmatrix} 4 & 1 & 2 & 1 \\ -3 & -1 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 4 & 1 \\ 1 & 1 & 6 & 0 \end{bmatrix}$

$= \begin{bmatrix} 7 & 0 & 6 & 2 \\ -2 & 0 & 6 & 3 \end{bmatrix}$

$A-B = \begin{bmatrix} 1 & 2 & -2 & 0 \\ -4 & -2 & -6 & 3 \end{bmatrix}$

$2A-3B = \begin{bmatrix} 8 & 2 & 4 & 2 \\ -6 & -2 & 0 & 6 \end{bmatrix} - \begin{bmatrix} 9 & -3 & 12 & 3 \\ 3 & 3 & 18 & 0 \end{bmatrix}$

$= \begin{bmatrix} -1 & -10 & -8 & -1 \\ -9 & -5 & -18 & 6 \end{bmatrix}$

3 $A^T = \begin{bmatrix} 4 & -3 \\ 1 & -1 \\ 2 & 0 \\ 1 & 3 \end{bmatrix}$, $B^T = \begin{bmatrix} 3 & 1 \\ -1 & 1 \\ 4 & 6 \\ 1 & 0 \end{bmatrix}$

$2A^T = \begin{bmatrix} 8 & -6 \\ 2 & -2 \\ 4 & 0 \\ 2 & 6 \end{bmatrix}$

$A^T+B^T = \begin{bmatrix} 7 & -2 \\ 0 & 0 \\ 6 & 6 \\ 2 & 3 \end{bmatrix}$

$3B^T = \begin{bmatrix} 9 & 3 \\ -3 & 3 \\ 12 & 18 \\ 3 & 0 \end{bmatrix}$

4 $C=D$

$\begin{bmatrix} x-1 & 3y \\ \frac{z}{2} & w-1 \\ q & p+1 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 2 & 0 \\ -1 & 2 \end{bmatrix} \Rightarrow$

$x-1=1 \Rightarrow x=2$

$3y=6 \Rightarrow y=2$

$\frac{z}{2}=2 \Rightarrow z=4$

$w-1=0 \Rightarrow w=1$

$p+1=2 \Rightarrow p=1$

$q=-1 \Rightarrow q=-1$

Example

Let $A = \begin{bmatrix} 1 & -3 \\ 2 & 1 \\ 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & -4 \\ 3 & 0 & -1 \end{bmatrix}$

$C = \begin{bmatrix} 2 & 3 \\ 7 & 1 \end{bmatrix}$

① Find AB ② Find BA ③ Is $AB = BA$

④ If its possible find CA and AC.

Solu.:

① $A \cdot B = \begin{bmatrix} 1 & -3 \\ 2 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -4 \\ 3 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -8 & 2 & -1 \\ 5 & 4 & -9 \\ 4 & 8 & -16 \end{bmatrix}$

② $B \cdot A = \begin{bmatrix} 1 & 2 & -4 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 2 & 1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} -11 & -1 \\ -1 & -9 \end{bmatrix}$

③ $AB \neq BA$

④ $CA \longrightarrow C_{2\times 2} \cdot A_{3\times 2}$, then CA is not defined

$2 \neq 3$

$AC = \begin{bmatrix} 1 & -3 \\ 2 & 1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 7 & 1 \end{bmatrix} = \begin{bmatrix} -19 & 0 \\ 11 & 7 \\ 8 & 12 \end{bmatrix}$

3×2 2×2

Submatrix

Any matrix obtained by omitting some rows and columns from a given $m \times n$ matrix A is called a submatrix of A.

Example

Submatrices of a matrix

The matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

Contains: three 2×2 submatrices, namely

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}, \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

and two 1×3 submatrices (the two row vectors)

$$[a_{11} \quad a_{12} \quad a_{13}], \quad [a_{21} \quad a_{22} \quad a_{23}]$$

It also contains three 2×1 submatrices (the column vectors)

$$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}, \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

and six 1×2 submatrices:

$$[a_{11} \quad a_{12}], \quad [a_{11} \quad a_{13}], \quad [a_{12} \quad a_{13}]$$

$$[a_{21} \quad a_{22}], \quad [a_{21} \quad a_{23}], \quad [a_{22} \quad a_{23}]$$

and six 1×1 submatrices:

$$[a_{11}], [a_{12}], [a_{13}], [a_{21}], [a_{22}], [a_{23}].$$