Determinants المحددات

Defn.

Let A be $(n \times n)$ matrix (Square matrix), the determinant of A is denoted by ($\det A$) or $|A|$ and is defined by:

(1) $\det(a) = |a| = a$ i.e $|-2| = -2$ --- (9)

(2) $$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

(3) $$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{vmatrix} \overset{+}{a_{11}} & \overset{-}{a_{12}} & \overset{+}{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example: Evalue $\begin{vmatrix} 2 & -4 \\ -5 & 3 \end{vmatrix}$

Solu. $$\begin{vmatrix} 2 & -4 \\ -5 & 3 \end{vmatrix} = (2)(3) - (-5)(-4) = 6 - 20 = -14$$

Example: Evaluate

$$\begin{vmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 2\begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix} - \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix}$$

$$= 2\left[(-1)(1) - (-2)(3)\right] - \left[(3)(1) - (-2)(2)\right]$$
$$+ 3\left[(3)(3) - (-1)(2)\right]$$

$$= 2(-1+6) - (3+4) + 3(9+2)$$
$$= 10 - 12 + 33 = 36$$

Minors and Cofactor :

Each element a_{jk} of a determinant A has minors M_{jk} which is a determinant obtained by deleting the jth row and k^{th} column.

EX. The third order determinant $\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

The minor of a_{12} is the second order determinant $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31}$

The Cofactor A_{jk} is defined as :

$$A_{jk} = (-1)^{j+k} M_{jk} \qquad \text{--- (10)}$$

Example :

The Cofactor of a_{12} is

$$A_{12} = (-1)^{1+2} M_{12}$$

$$= -\begin{vmatrix} a_{21} & a_{33} \\ a_{31} & a_{33} \end{vmatrix}$$

OR The sign array may be written as

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

now, the determinant

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

may be evaluated by $\det A = a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13}$

and may be performed in five order ways using the elements of the other two rows and three columns.

Example:

Determine the minor M_{23} of $\begin{bmatrix} 1 & 2 & -1 \\ 4 & 3 & -4 \\ -2 & 0 & -3 \end{bmatrix}$

Solu.

First delete the 2nd row and 3rd column which yeilds the matrix $\begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}$

then M_{23} is the determinant of this matrix

$$M_{23} = \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} = (1)(0) - (-2)(2) = 4.$$

Example

Evaluate $\begin{vmatrix} -3 & 2 & 1 \\ -5 & 0 & 6 \\ -2 & -1 & 3 \end{vmatrix}$

Solu.

choose the row or column where there is a number 0.

$$= -(-5)\begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} + 0 - 6\begin{vmatrix} -3 & 2 \\ -2 & -1 \end{vmatrix}$$

$$= 5[6-(-1)] - 6[(-3)(-1) - (-2)(2)]$$

$$= 35 - 42 = -7.$$

Properties of Determinants

(a) If two rows or columns are identical, the determinant is zero, for example

$$\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0 \quad , \quad \begin{vmatrix} -1 & -1 \\ -3 & -3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 & 3 \\ 0 & 4 & 3 \\ 2 & 1 & 3 \end{vmatrix} = 0 \quad , \quad \begin{vmatrix} 2 & 1 & 2 \\ 2 & 3 & 2 \\ 2 & 4 & 2 \end{vmatrix} = 0$$

(b) If all entries of a single row or column are zero, then the determinant is zero

for example:

$$\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} = 0 \quad , \quad \begin{vmatrix} 3 & 0 \\ 2 & 0 \end{vmatrix} = 0$$

(c) Interchanging any two rows or two columns alters the det. by a factor of -1

for example:

$$\begin{vmatrix} 2 & 6 & 4 \\ 1 & 3 & 0 \\ -1 & 0 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 12$$

(d) $|A| = |A^T|$

(e) If A is a diagonal matrix or triangular matrix, the det A is the product of the elements on the main diagonal.

EX.

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \quad , \quad \begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} = -2 \quad , \quad \begin{vmatrix} 2 & 6 & -6 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{vmatrix} = (2)(3)(-1) = -6 .$$

f. If each element of some rows or columns of a matrix A is a multiplied by a constant k then the determinant is multiplied by k.

Ex.

$$\begin{vmatrix} 3 & 3 \\ 5 & -1 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} = (3)\left[(1)(-1) - (5)(1)\right] = 3(-1-5) = -18$$

g. If we interchange the ith and the jth row of the matrix A the the value of the determinant is change by a factor of -1

$$54 = \begin{vmatrix} 2 & 3 & 1 \\ 4 & -2 & 0 \\ 5 & 7 & -1 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{vmatrix} 5 & 7 & -1 \\ 4 & -2 & 0 \\ 2 & 3 & 1 \end{vmatrix} = -54$$

h. If we add k times the jth row to the ith row and put it in place of the ith row, then the determinant still same.

$$-12 = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \\ 3 & 2 & 5 \end{vmatrix} \xrightarrow[R_3 \leftrightarrow R_2 + R_3]{R_1 \leftrightarrow -R_2 + R_1} \begin{vmatrix} -3 & 2 & -2 \\ 4 & 0 & 1 \\ 7 & 2 & 6 \end{vmatrix} = -12$$

Ex. For the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

$$a_{11} = \begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} = 6 \quad , \quad a_{12} = \begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} = -2$$

$$a_{13} = \begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} = -3 \quad , \quad a_{21} = 1 \quad , \quad a_{22} = -5$$

$$a_{23} = 3 \qquad a_{31} = -5 \quad , \quad a_{32} = 4 \quad , \quad a_{33} = -1$$

and $\operatorname{adj} A = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ \end{bmatrix} \Rightarrow \operatorname{adj} A \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & 1 \end{bmatrix}$

The Adjoint or Adjugate Matrix

Let $A=[a_{ij}]$ be an n-square matrix and a_{ij} be the cofactor of a_{ij}, then ~~by~~ definition

$$\text{adjoint } A \equiv \text{adj} A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix} \quad \text{---(11}$$

that is, the adjoint of A is obtained by replacing each element of A by its cofactor and transposing.

Ex.

Let $A = \begin{bmatrix} 4 & 2 & 1 \\ 3 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, find its adjoint

Solu.

$$a_{11} = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1 \quad , \quad a_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} = -6$$

$$a_{13} = (-1)^{4} \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3 \quad \text{--- etc}$$

$$\text{adj} A = \begin{bmatrix} -1 & -6 & 3 \\ -3 & 8 & -4 \\ 2 & -1 & -6 \end{bmatrix}^T = \begin{bmatrix} -1 & -3 & 2 \\ -6 & 8 & -1 \\ 3 & -4 & -6 \end{bmatrix}$$

Inverse Matrix

if A is a square matrix of order n, then A has an inverse if and only if there is a square matrix B of order n such that $AB = I_n$.

B is called inverse of A and written A^{-1}

$\therefore \quad AA^{-1} = I_n$ and $A^{-1}A = I_n$ --- (12)

A square matrix has an inverse if its determinant $\neq 0$

To find the inverse of a matrix whose (determinant is not zero).

(a) Construct the matrix ~~whose~~ of cofactor of A

$$\text{cof}(A) = C_{ij}$$

(b) Construct the transposed matrix of cofactors called (the adjoint of A)

$$\text{adj}(A) = (\text{cof}A)^T$$

(c) Then $A^{-1} = \dfrac{\text{adj}(A)}{\det A}$

(d) To check your answer $A \cdot A^{-1} = I$

Example: Find the inverse of the matrix $A = \begin{bmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \\ 3 & -1 & -1 \end{bmatrix}$

Solu.

$$C_{11} = (-1)^{1+1} A_{11} = \begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} = -2+3 = 1$$

$$C_{12} = (-1)^{1+2} A_{12} = -\begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix} = 10$$

$$C_{13} = (-1)^{1+3} A_{13} = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -1-6 = -7$$

$$C_{21} = (-1)^{2+1} A_{21} = \begin{vmatrix} 3 & -4 \\ -1 & -1 \end{vmatrix} = -(-3-4) = 7$$

$$C_{22} = (-1)^{2+2} A_{22} = \begin{vmatrix} 2 & -4 \\ 3 & -1 \end{vmatrix} = -2+12 = 10$$

$$C_{23} = (-1)^{2+3} A_{23} = \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -(-2-9) = 11$$

$$C_{31} = (-1)^{3+1} A_{31} = \begin{vmatrix} 3 & -4 \\ 2 & 3 \end{vmatrix} = 9+8 = 17$$

$$C_{32} = (-1)^{3+2} A_{32} = -\begin{vmatrix} 2 & -4 \\ 1 & 3 \end{vmatrix} = -(6+4) = -10$$

$$C_{33} = (-1)^{3+3} A_{33} = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 4-3 = 1$$

$$Cof(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 1 & 10 & -7 \\ 7 & 10 & 11 \\ 17 & -10 & 1 \end{bmatrix}$$

$$\text{adj}\, A = (\text{cof}\, A)^T = \begin{bmatrix} 1 & 7 & 17 \\ 10 & 10 & -10 \\ -7 & 11 & 1 \end{bmatrix}$$

$$\det(A) = |A| = \begin{vmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \\ 3 & -1 & -1 \end{vmatrix} = 60$$

$$A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{60}\begin{bmatrix} 1 & 7 & 17 \\ 10 & 10 & -10 \\ -7 & 11 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{60} & \frac{7}{60} & \frac{17}{60} \\ 1/6 & 1/6 & -1/6 \\ \frac{-7}{60} & \frac{11}{60} & \frac{1}{60} \end{bmatrix}$$

Linear Equations المعادلات الخطية

Consider a system of Linear equations in three unknowns x, y, z

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}\right\} \quad \dots (13)$$

In matrix notation the system of Linear eq.'s may be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \dots (14)$$

The above theorem called <u>Grammer's Rule</u>.

To solve the above theorem we put :

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \quad D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

if $D \neq 0$, then the system has a unique solution which is

$$x = \frac{D_1}{D}, \quad y = \frac{D_2}{D}, \quad z = \frac{D_3}{D}.$$

EX.: Use Grammer's Rule to solve

$$5x - 2y = -1$$
$$2x + 3y = 3$$

Solu.

$$\begin{bmatrix} 5 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$D = \begin{vmatrix} 5 & -2 \\ 2 & 3 \end{vmatrix} = 15 + 4 = 19$$

$$D_1 = \begin{vmatrix} -1 & -2 \\ 3 & 3 \end{vmatrix} = -3 + 6 = 3 \quad ; \quad D_2 = \begin{vmatrix} 5 & -1 \\ 2 & 3 \end{vmatrix} = 15 + 2 = 17$$

$$\therefore x = \frac{D_1}{D} = \frac{3}{19} \quad , \quad y = \frac{D_2}{D} = \frac{17}{19} .$$

EX. Use Grammer's Rule to solve

$$x + \quad + 2z = 6$$
$$-3x + 4y + 6z = 30$$
$$-x - 2y + 3z = 8$$

Solu.

$$D = \begin{vmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix} = +11 \quad , \quad D_1 = \begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{vmatrix} = -10$$

$$D_2 = \begin{vmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{vmatrix} = 18 \quad , \quad D_3 = \begin{vmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{vmatrix} = 38$$

$$x = \frac{D_1}{D} = \frac{-10}{11} \quad , \quad y = \frac{D_2}{D} = \frac{18}{11} \quad , \quad z = \frac{D_3}{D} = \frac{38}{11}$$

Solution of a System of Linear Equations by: (Inverse Matrix Method)

Consider the system of n Linear equations in n unknowns X_i,

$$AX = B \qquad (15)$$

where A is an nxn matrix $[a_{jk}]$, X is the Column matrix $[X_i]$ and B is the Column matrix $[b_i]$. If A is non-singular ($\det A \neq 0$), then A^{-1} exists and equation(15) may be written

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

and

$$X = A^{-1}B \qquad (16)$$

Equation(16) is then the formal solution of equation(15).

The solution(16) may be expressed in terms of determinants and cofactors. As illustration consider the case n=3, then

$$X = \frac{1}{\det A}(\operatorname{adj} A)B = \frac{1}{\det A}\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$X = \frac{1}{\det A}\begin{bmatrix} b_1a_{11} + b_2a_{21} + b_3a_{31} \\ b_1a_{12} + b_2a_{22} + b_3a_{32} \\ b_1a_{13} + b_2a_{23} + b_3a_{33} \end{bmatrix} \qquad (17)$$

<u>Example</u> Solve the System by (Inverse Matrix Method):

$$2X_1 + X_2 - X_3 = 4$$

$$X_1 - 2X_2 + X_3 = 10$$

$$-3X_1 \quad -2X_3 = 9$$

<u>Solu.</u>

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ -3 & 0 & -2 \end{bmatrix}, \quad X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ -10 \\ 9 \end{bmatrix}$$

Since $\det A = (-3)(-1) + (-2)(-5) = 13$,

$$A^{-1} = \frac{1}{13}\begin{bmatrix} 4 & 2 & -1 \\ -1 & -7 & -3 \\ -6 & -3 & -5 \end{bmatrix}$$

and $$X = \frac{1}{3}\begin{bmatrix} 4 & 2 & -1 \\ -1 & -7 & -8 \\ -6 & -3 & -5 \end{bmatrix}\begin{bmatrix} 4 \\ -10 \\ 9 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} -13 \\ 39 \\ -39 \end{bmatrix}$$

Hence $X_1 = -1$, $X_2 = 3$, $X_3 = -3$

<u>Exercises</u>

① Solve by the Grammer's Rule

(a)
$$2X_1 - 5X_2 = -2$$
$$4X_1 + 6X_2 = 1$$

(b)
$$x + 2y + z = 3$$
$$2x + y - z = 0$$
$$x - y + z = 0$$

② Find the inverse matrix of A

(a) when $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

(b) when $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

③ If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & -2 \\ 1 & -5 \\ 4 & 3 \end{bmatrix}$ Find $D = \begin{bmatrix} p & q \\ r & s \\ t & u \end{bmatrix}$

such that $A + B - D = 0$.