Partial Derivatives

Functions of Two or More Variables

Definition:

A function of two real variables x and y is a rule that assigns a single real number $f(x,y)$ to each point (x,y) in some set D of the xy-plane.

Definition:

A function of three variables x, y and z is a rule that assigns a single real number $f(x,y,z)$ to each point (x,y,z) in some set D of three-dimensional space.

The set D is the domain of the function, it is the set of points at which the function is defined.

Example(1):

Let $f(x,y) = 3x^2\sqrt{y} - 1$. Find $f(1,4)$, $f(0,9)$, $f(t^2,t)$, $f(ab,9b)$, and the domain of f.

Solution: By substitution

$$f(1,4) = 3(1)^2\sqrt{4} - 1 = 5.$$

$$f(0,9) = 3(0)^2\sqrt{9} - 1 = -1.$$

$$f(t^2,t) = 3(t^2)^2\sqrt{t} - 1 = 3t^4\sqrt{t} - 1.$$

$$f(ab,9b) = 3(ab)^2\sqrt{9b} - 1 = 9a^2b^2\sqrt{b} - 1.$$

(The domain of f)

Example(2): If $f(x,y,z)=\sqrt{1-x^2-y^2-z^2}$. Find $f(0,\frac{1}{2},-\frac{1}{2})$ and the domain of the function f.

Solution:

$$f(0,\tfrac{1}{2},-\tfrac{1}{2})=\sqrt{1-(0)^2-(\tfrac{1}{2})^2-(-\tfrac{1}{2})^2}=\sqrt{\tfrac{1}{2}}.$$

we must have $1-x^2-y^2-z^2 \geq 0$ or $x^2+y^2+z^2 \leq 1$

thus we see that the domain of f consists of all points on or within the sphere $x^2+y^2+z^2=1$.

Limits and Continuity النهايات والاستمرارية

Limits

Definition: let f be a function of two variables, and suppose that f is defined at each point in some circular region centered at (x_0,y_0) except possibly at (x_0,y_0) it self.

We shall write:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=L$$ if given any number $\epsilon>0$, we can find a number $\delta>0$ such that $f(x,y)$ satisfies

$|f(x,y)-L|<\epsilon$, whenever the distance between (x,y) and (x_0,y_0) satisfies:

$$0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta.$$

<u>Definition</u> :

Let f be a function of three variables, and suppose that f is defined at each point in some spherical region centered at (x_0, y_0, z_0) except possibly at (x_0, y_0, z_0) itself.

We shall write

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = L \quad , \text{ if given any number } \epsilon > 0,$$

we can find a number $\delta > 0$ such that $f(x,y,z)$ satisfies :

$$|f(x,y,z) - L| < \epsilon \qquad \text{whenever the distance between } (x,y,z)$$

and (x_0, y_0, z_0) satisfies :

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} < \delta .$$

<u>Theorem</u> (Properties of Limits)

If $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L_1$ and $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = L_2$

then,

(a) $\lim [c\, f(x,y)] = c \lim f(x,y) = c\, L_1$

(where c is a constant).

(b) $\lim [f(x,y) \pm g(x,y)] = \lim f(x,y) \pm \lim g(x,y)$
$= L_1 \pm L_2$.

ⓒ $\lim [f(x,y) \cdot g(x,y)] = L_1 L_2$.

ⓓ $\lim \dfrac{f(x,y)}{g(x,y)} = \dfrac{\lim f(x,y)}{\lim g(x,y)} = \dfrac{L_1}{L_2}$ $(L_2 \neq 0)$.

i.e., $\lim$ represents $\lim\limits_{(x,y)\to(x_0,y_0)}$.

Similarly, for functions of three variables.

Continuity الاستمرارية

Definition: A function f of two variables is called continuous at the point (x_0,y_0) if:

1. $f(x_0,y_0)$ is defined .
2. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists .
3. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.

Definition: A function f of three variables is called continuous at a point (x_0,y_0,z_0) if:

1. $f(x_0,y_0,z_0)$ is defined.
2. $\lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z)$ exists.
3. $\lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = f(x_0,y_0,z_0)$.

Example :

The following functions are Continuous :

(1) $f(x,y) = 3x^2y^5$.

(2) $f(x,y) = (x^4y^5)^{1/3}$.

(3) $f(x,y) = e^{xy}$.

(4) $f(x,y) = \cosh(x^3y)$.

Example :

The function $f(x,y) = \dfrac{x^3y^2}{1-xy}$ is Continuous except where $1-xy=0$. Thus $f(x,y)$ is continuous everywhere except on the hyperbola $xy=1 \Rightarrow y=\frac{1}{x}$

Example : Evaluate $\displaystyle\lim_{(x,y)\to(-1,2)} \frac{xy}{x^2+y^2}$.

Solution: The function is Continuous at $(-1,2)$ (H.W).

Thus

$$\lim_{(x,y)\to(-1,2)} \frac{xy}{x^2+y^2} = \frac{(-1)(2)}{(-1)^2+(2)^2} = \frac{-2}{5} .$$

Example : Let $f(x,y) = \dfrac{xy}{x^2+y^2}$. Find the limit of $f(x,y)$ as $(x,y) \to (0,0)$ along

(a) The x-axis , (b) The y-axis , (c) The parabola $y=x^2$

(d) The Line $y=x$.

Solution:

(a) If $(x,y) \to (0,0)$ along the x-axis, then $y=0$, so

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} f(x,0) = \lim_{(x,y)\to(0,0)} \frac{0}{x^2} = \lim_{(x,y)\to(0,0)} 0 = 0.$$

(along $y=0$)

(b) If $(x,y) \to (0,0)$ along the y-axis, then $x=0$, so

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} f(0,y) = \lim_{(x,y)\to(0,0)} \frac{0}{y^2} = 0.$$

(along $x=0$)

c) If $(x,y) \to (0,0)$ along $y=x^2$, then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} f(x,x^2) = \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+x^4}$$

(along $y=x^2$)

$$= \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2(1+x^2)} = \lim_{(x,y)\to(0,0)} \frac{x}{1+x^2} = \frac{0}{1+(0)^2} = 0.$$

(d) If $(x,y) \to (0,0)$ along $y=x$, then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} f(x,x) = \lim_{(x,y)\to(0,0)} \frac{x^2}{2x^2} = \frac{1}{2}.$$

(along $y=x$)

The limit of the function $f(x,y)$ does not exist. Because the limit does not have the same value along all paths to $(0,0)$.

Partial Derivatives

Geometric meaning of partial derivatives

لو كان لدينا دالة مثل $y=f(x)$

$$y = f(x) = x^2+1$$
$$y' = f'(x) = 2x$$

فلو اردنا ايجاد ميل المماس للمنحني $y = x^2+1$ عند النقطة $x=1$ مثلاً

فيكون

$$f'(1) = 2(1) = 2$$

وكما يلي

وان علاقة مثل $y=f(x)$ تمثل منحني مرسوم في 2-D لها x, y اما اذا كان لدينا علاقة مثل $z=f(x,y)$ فانها تمثل سطح مرسوم في 3-D لها x,y,z وكما يلي :

واذا اخذنا نقطة معينة على السطح مثل $P_0(x_0,y_0,z_0)$ وقمنا برسم مماس للسطح عند تلك النقطة P_0 باتجاه يوازي محور y فسوف نحصل على الشكل التالي

ان ميل المماس هذا عندئذ يسمى بالمشتقة الجزئية لـ f بالنسبة الى y عند النقطة (x_0,y_0) ويكتب كما يلي :

$$\left.\frac{\partial f}{\partial y}\right|_{x_0,y_0} \quad \text{or} \quad \left.\frac{\partial z}{\partial y}\right|_{x_0,y_0} \quad \text{or} \quad f_y(x_0,y_0) \Rightarrow$$

Partial derivative of f with respect to y at (x_0,y_0).

والمعنى الرياضي لـ $f_y(x_0,y_0)$ هو ميل المماس للسطح $z=f(x,y)$ عند النقطة P_0 عندما يكون x ثابتاً .

ونفس المفهوم يمكن تصوره اذا رسمنا مماس للسطح $z=f(x,y)$ عند النقطة $P_0(x_0,y_0,z_0)$ بحيث يكون المماس باتجاه يوازي محور x وكما يلي :

فيكون عندئذ ميل المماس للسطح هو

The partial derivative of f with respect to x at (x_0, y_0)

$$\left.\frac{\partial f}{\partial x}\right|_{x_0, y_0} \quad \text{or} \quad \left.\frac{\partial z}{\partial x}\right|_{x_0, y_0} \quad \text{or} \quad f_x(x_0, y_0).$$

Partial derivatives of functions with two variables

تعريف

Defn. If (x_0, y_0) is a point in the domain of a function $z = f(x, y)$ then the derivative of $z = f(x, y_0)$ with respect to x at $x = x_0$ is defined as

$$\left.\frac{d}{dx} f(x, y_0)\right|_{x = x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

, provided that the limit exist.

This limit is called the partial derivative of f with respect to x at the point (x_0, y_0) and denoted by f_x, z_x, $\frac{\partial f}{\partial x}$, $\frac{\partial z}{\partial x}$.

This definition show that $\frac{\partial f}{\partial x}$ is an ordinary derivative of f with respect to x holding y fixed.

Similarly

$$\left.\frac{d}{dy} f(x_0, y)\right|_{x_0, y_0} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$, provided that the limit exist.

is called the partial derivative of f with respect to y at point (x_0, y_0) and denoted by f_y, z_y, $\frac{\partial f}{\partial y}$, $\frac{\partial z}{\partial y}$.

EX.(1)

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ if $f(x,y) = 2 - x^2 - 3y^4 - \frac{x^3}{y}$.

Solu.

$$\frac{\partial f}{\partial x} = -2x - \frac{3x^2}{y}, \quad \frac{\partial f}{\partial y} = -12y^3 + \frac{x^3}{y^2}.$$

EX.(2) If $z = e^{2x} \ln(x^2 + 2y^2 - x\sin y + 1)$. Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

Solu.

$$\frac{\partial z}{\partial x} = e^{2x} \frac{(2x - \sin y)}{(x^2 + 2y^2 - x\sin y + 1)} + 2\ln(x^2 + 2y^2 - x\sin y + 1) \cdot e^{2x}$$

$$\frac{\partial z}{\partial y} = e^{2x} \frac{(4y - x\cos y)}{(x^2 + 2y^2 - x\sin y + 1)}.$$

EX.(3) Given $f(x,y) = e^{xy} \cdot \cos^2 x \cdot \sin 2y$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

solu.

$$\frac{\partial f}{\partial x} = \sin(2y)\left[-2e^{xy}\cos x \sin x + y e^{xy}\cos^2 x\right].$$

$$\frac{\partial f}{\partial y} = \cos^2 x \left[2e^{xy}\cos 2y + x e^{xy}\sin 2y\right].$$

Partial derivatives of functions with more than two Variables

The definition of the partial derivatives of functions with more than two variables are like the definition for functions of two variables. They are ordinary derivatives with respect to one variable taken, while all the other independent variables are regared as constants.

Thus if $w = f(x_1, x_2, \dots, x_n)$ then we may have n partial derivati denoted by f_{x_i}, $\frac{\partial f}{\partial x_i}$, $\frac{\partial w}{\partial x_i}$, $i = 1, 2, \dots, n$.

EX.(4) Let $f(x,y,z) = x z^2 \tan^{-1}\left(\frac{y}{x}\right)$. Find f_x, f_y, f_z.

Solu.

$$f_x = z^2 \left[\frac{x\left(-\frac{y}{x^2}\right)}{1+\left(\frac{y}{x}\right)^2} + \tan^{-1}\left(\frac{y}{x}\right) \right], \quad f_y = \frac{x z^2 \left(\frac{1}{x}\right)}{1+\left(\frac{y}{x}\right)^2}$$

$$f_z = 2xz \tan^{-1}\left(\frac{y}{x}\right).$$

EX.(5) If $w = (xy)^{\sin z}$. Find $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$, $\frac{\partial w}{\partial z}$

Solu.

$$\frac{\partial w}{\partial x} = y \sin z \, (xy)^{\sin z - 1}, \quad \frac{\partial w}{\partial y} = x \sin z \, (xy)^{\sin z - 1}$$

$$\frac{\partial w}{\partial z} = (xy)^{\sin z} \ln(xy) \cos z.$$

Ex. (6) If z is a differentiable function of x and y which satisfy the equation $x^3+y^3+z^3+3x^2\sin y\tan z=5$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

Solu.

$$3x^2+3z^2\frac{\partial z}{\partial x}+3\sin y\left(x^2\sec^2 z\frac{\partial z}{\partial x}+2x\tan z\right)=0$$

$$\Rightarrow \frac{\partial z}{\partial x}=-\frac{3x^2+6x\sin y\tan z}{3z^2+3x^2\sin y\sec^2 z}.$$

$$3y^2+3z^2\frac{\partial z}{\partial y}+3x^2\left(\sin y\sec^2 z\frac{\partial z}{\partial y}+\tan z\cos y\right)=0$$

$$3y^2+\left(3z^2+3x^2\sin y\sec^2 z\right)\frac{\partial z}{\partial y}+3x^2\tan z\cos y=0$$

$$\Rightarrow \frac{\partial z}{\partial y}=-\frac{3y^2+3x^2\cos y\tan z}{3z^2+3x^2\sin y\sec^2 z}.$$

Chain Rule قاعدة السلسلة

Chain Rule For Functions of Two Variables Defined Along Paths

Remember, when we had a differentiable function of a single variable say $y=f(x)$, with x itself is a differentiable function of t, say $x=x(t)$.

We found that $y=f[x(t)]$ was a differentiable function of t with

$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$$

For example

If $y = x^2+1$ and $x = \tan^{-1} t$, then

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = 2x \cdot \frac{1}{1+t^2} = \frac{2\tan^{-1} t}{1+t^2}.$$

or one may write $y = (\tan^{-1} t)^2 + 1 \Rightarrow \frac{dy}{dt} = 2\tan^{-1} t \cdot \frac{1}{1+t^2}$

Now, consider the following example

$$x^2 \sin y + y^3 e^{2x} = 5 \quad \text{identical to} \quad f(x,y) = C$$

where $f(x,y) = x^2 \sin y + y^3 e^{2x}$ and $C = 5$

$$\frac{\partial f}{\partial x} = 2x \sin y + 2y^3 e^{2x}, \quad \frac{\partial f}{\partial y} = x^2 \cos y + 3y^2 e^{2x}$$

Let us differentiable the eq. $x^2 \sin y + y^3 e^{2x} = 5$ implicitly

$$x^2 \cos y \frac{dy}{dx} + 2x \sin y + 2y^3 e^{2x} + 3y^2 e^{2x} \frac{dy}{dx} = 0, \quad df = 0$$

$$x^2 \cos y \, dy + 2x \sin y \, dx + 2y^3 e^{2x} dx + 3y^2 e^{2x} dy = 0$$

$$(2x \sin y + 2y^3 e^{2x}) dx + (x^2 \cos y + 3y^2 e^{2x}) dy = 0$$

we see that

$$df = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy$$

The analagous formula for the function $z = f(x,y)$ of two variables that has partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, we have

$$\boxed{df = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy} \quad \text{---- (1)}$$

when $x = x(t)$ and $y = y(t)$ are differentiable functions of t
then $z = f(x,y) = f[x(t), y(t)]$ is a differentiable function of t

and

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}} \quad \cdots (2)$$

<u>Ex.(7)</u> Given $z = f(x,y) = x^2y^3$, where $x = \cos t$, $y = \sin t$

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = 2xy^3(-\sin t) + 3x^2y^2(\cos t)$$

$$= 2\cos t \sin^3 t(-\sin t) + 3\cos^2 t \sin^2 t(\cos t)$$

$$= -2\cos t \sin^4 t + 3\cos^3 t \sin^2 t.$$

<u>Note</u> We may check this result as

$$f(x,y) = f[x(t), y(t)] = \cos^2 t \sin^3 t$$

$$\Rightarrow \frac{df}{dt} = 3\cos^2 t \sin^2 t \cos t + 2\sin^3 t \cos t(-\sin t)$$

$$= 3\cos^3 t \sin^2 t - 2\sin^4 t \cos t$$

Chain Rule For Functions of More Than Two Variables Defined Along Paths :

If $w = f(x_1, x_2, \cdots, x_n)$ of n variables that has partial derivatives $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n}$ if $x_i = x_i(t)$ $i = 1, 2, \cdots, n$ are differentiable functions of t, then chain rule for function of more than two variables defined as

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\cdot\frac{dx_2}{dt} + \cdots + \frac{\partial f}{\partial x_n}\cdot\frac{dx_n}{dt}} \quad \cdots (3)$$

<u>Ex.(8)</u> Let $w = xy\sin z$ where $x = \cos t$, $y = \sin t$, $z = 1 + t^2$

Find $\frac{dw}{dt}$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$$= y\sin z(-\sin t) + x\sin z(\cos t) + xy\cos z\,(2t)$$

$$= \sin t\,\sin(1+t^2)(-\sin t) + \cos t\,\sin(1+t^2)\cos t + \cos t\,\sin t\,\cos(1+t^2)(2t)$$

$$= -\sin^2 t\,\sin(1+t^2) + \cos^2 t\,\sin(1+t^2) + 2t\,\cos t\,\sin t\,\cos(1+t^2).$$

" Chain Rule For Functions of Two Variables Defined on Surfaces "

If $z = f(x,y)$ has partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and if $x = x(r,s)$, $y = y(r,s)$ also have partial derivatives then $z = f[x(r,s), y(r,s)]$ and

$$\boxed{\frac{\partial z}{\partial r} = \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}} \quad \ldots(4)$$

$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}} \quad \ldots(5)$$

Ex.(9) Find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial s}$ if $z = f(x,y) = x^2 + y^2$ where $x = r + e^s$ and $y = \ln s$.

Solu.

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r} = 2x\,(1) + 2y\,(0) = 2x = 2(r + e^s).$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} = 2x\,(e^s) + 2y\left(\frac{1}{s}\right) = 2(r + e^s)\,e^s + 2(\ln s)\frac{1}{s}$$

$$= 2re^s + 2e^{2s} + \frac{2\ln s}{s}.$$

"Chain Rule For Functions of Three Variables Defined on Surfaces"

If $w = f(x,y,z)$ has partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ and if $x = x(r,s)$, $y = y(r,s)$, $z = z(r,s)$ also have partial derivatives then

$w = f[x(r,s), y(r,s), z(r,z)]$, and

$$\boxed{\frac{\partial w}{\partial r} = \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial r}} \quad \ldots (6)$$

$$\boxed{\frac{\partial w}{\partial s} = \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial s}} \quad \ldots (7)$$

EX.(10): Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ if $w = f(x,y,z) = x + 2y + z^2$ where

$x = \frac{r}{s}$, $y = r^2 + e^s$, $z = 2r$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r} = (1)\left(\frac{1}{s}\right) + 2(2r) + 2z(2)$$

$$= \frac{1}{s} + 4r + 4(2r) = \frac{1}{s} + 12r .$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s} = (1)\left(-\frac{r}{s^2}\right) + 2e^s + 0$$

$$= \frac{-r}{s^2} + 2e^s .$$

In general chain Rule for functions of many variables

If $w = f(x_1, x_2, \dots, x_n)$ has partial derivatives $\frac{\partial f}{\partial x_i}$, $i = 1, 2, \dots, n$ and each variable x_i $(i = 1, 2, \dots, n)$ is related to another finite set of variables, say $t_1, t_2, \dots, t_m$. That is,

$x_i = x_i(t_1, t_2, \dots, t_m)$, $i = 1, 2, \dots, n$ then

$$w = f[x_1(t_1, t_2, \dots, t_m), x_2(t_1, t_2, \dots, t_m), \dots, x_n(t_1, t_2, \dots, t_m)]$$

and العلاقة العامة

$$\boxed{\frac{\partial w}{\partial t_j} = \frac{\partial f}{\partial t_j} = \frac{\partial w}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_1} + \frac{\partial w}{\partial x_2}\cdot\frac{\partial x_2}{\partial t_2} + \dots + \frac{\partial w}{\partial x_n}\cdot\frac{\partial x_n}{\partial t_i} \quad \dots (8)}$$

$j = 1, 2, \dots, m$

EX.(11) If $w = (x^2 + y^2 + z^2)^{1/2}$, where $x = e^r \cos\theta$, $y = e^r \sin\theta$, $z = re^\theta$.
Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$ by chain rule and check your result by different method.

Solu. :

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r} = \frac{x e^{r}\cos\theta + y e^{r}\sin\theta + z e^{\theta}}{(x^2+y^2+z^2)^{1/2}}$$

but $x^2+y^2+z^2 = e^{2r}\cos^2\theta + e^{2r}\sin^2\theta + r^2 e^{2\theta} = e^{2r} + r^2 e^{2\theta}$

$$\therefore \frac{\partial w}{\partial r} = \frac{e^{2r}\cos^2\theta + e^{2r}\sin^2\theta + r e^{2\theta}}{(e^{2r}+r^2 e^{2\theta})^{1/2}} = \frac{e^{2r}+r e^{2\theta}}{(e^{2r}+r^2 e^{2\theta})^{1/2}}$$

by different method :

$$w = (e^{2r}+r^2 e^{2\theta})^{1/2} \Rightarrow \frac{\partial w}{\partial r} = \frac{1}{2}(e^{2r}+r^2 e^{2\theta})^{-\frac{1}{2}}[2e^{2r}+2r e^{2\theta}]$$

$$\therefore \frac{\partial w}{\partial r} = \frac{\cancel{2}(e^{2r}+r e^{2\theta})}{\cancel{2}(e^{2r}+r^2 e^{2\theta})^{1/2}}.$$

Similarly

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s} = \frac{r e^{2\theta}}{(e^{2r}+r^2 e^{2\theta})^{1/2}}.$$

EX.(12): Show that any differentiable function of the form $y = f(x)$ where $x = r + as$ (a is constant $\neq 0$) is a solution of the partial differentiable equation $\frac{\partial y}{\partial s} - a\frac{\partial y}{\partial r} = 0$.

Solu.

$$\frac{\partial y}{\partial r} = \frac{dy}{dx}\cdot\frac{\partial x}{\partial r} \quad \text{and} \quad \frac{\partial y}{\partial s} = \frac{dy}{dx}\cdot\frac{\partial x}{\partial s}$$

$$\frac{\partial y}{\partial s} - a\frac{\partial y}{\partial r} = \frac{dy}{dx}\cdot\frac{\partial x}{\partial s} - a\frac{dy}{dx}\cdot\frac{\partial x}{\partial r}$$

$$= \frac{dy}{dx}a - a\frac{dy}{dx}(1)$$

$$= a\frac{dy}{dx} - a\frac{dy}{dx} = 0$$

EX.(13): Let $z = f(t)$ where $t = \frac{x+y}{x}$. show that:

$$x^2\frac{\partial z}{\partial x} - y^2\frac{\partial z}{\partial y} = 0.$$

Solu.

$$\frac{\partial z}{\partial x} = \frac{dz}{dt}\cdot\frac{\partial t}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{dz}{dt}\cdot\frac{\partial t}{\partial y}$$

$$x^2\frac{\partial z}{\partial x} - y^2\frac{\partial z}{\partial y} = x^2\frac{dz}{dt}\cdot\frac{\partial t}{\partial x} - y^2\frac{dz}{dt}\cdot\frac{\partial t}{\partial y}$$

$$= x^2\frac{dz}{dt}\,\frac{xy-(x+y)y}{x^2y^2} - y^2\frac{dz}{dt}\,\frac{xy-(x+y)x}{x^2y^2}$$

$$= x^2\frac{dz}{dt}\left(-\frac{1}{x^2}\right) - y^2\frac{dz}{dt}\left(\frac{-1}{y^2}\right)$$

$$= -\frac{dz}{dt} + \frac{dz}{dt} = 0$$

EX.(14): Find $\frac{\partial w}{\partial x}$ at point $(x,y,z)=(1,1,1)$ if $w=\cos(uv)$ where

$$u = xyz \quad , \quad v = \frac{\pi}{4(x^2+y^2)} = \frac{\pi}{4}(x^2+y^2)^{-1}$$

Solu.

$$\frac{\partial w}{\partial x} = ?$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= -v\sin(uv)\cdot(yz) + \left(-u\sin(uv)\right)\left(\frac{-\pi x}{2(x^2+y^2)^2}\right)$$

$$\left.\frac{\partial w}{\partial x}\right|_{(x,y,z)=(1,1,1)} = -\frac{\pi}{8}\sin\left(\frac{\pi}{8}\right) + \sin\left(\frac{\pi}{8}\right)\cdot\frac{\pi}{8} = 0 .$$

Ex(15) If $w = f(u,v)$ where $u = x+y$ and $v = x-y$. Show that

$$\frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} = \left(\frac{\partial w}{\partial u}\right)^2 - \left(\frac{\partial w}{\partial v}\right)^2$$

Solu.

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial w}{\partial u}(1) + \frac{\partial w}{\partial v}(1)$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial w}{\partial u}(1) + \frac{\partial w}{\partial v}(-1)$$

$$\therefore \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} = \left(\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}\right)\left(\frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}\right)$$

$$= \left(\frac{\partial w}{\partial u}\right)^2 - \left(\frac{\partial w}{\partial v}\right)^2 .$$

Higher Order Partial Derivatives

المشتقات العليا

2nd order partial derivatives for functions with two independent variables

If $z = f(x,y)$ has partial derivatives $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$, then the 2nd order partial derivatives are denoted by:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}, \quad f_{xy} = \frac{\partial^2 f}{\partial y \partial x}, \quad f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$$

Note:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy} .$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial f_x}{\partial y} = f_{xy}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial f_y}{\partial x} = f_{yx} .$$

$\frac{\partial^2 f}{\partial y \partial x} = f_{xy}$ mean: differentiate partially with respect to x first and then with respect to y.

<u>EX.(16)</u> Let $f(x,y) = x\cos y + y e^x$. Find f_{xx}, f_{yy}, f_{yx}, f_{xy}

solu:

$$f_x = \frac{\partial f}{\partial x} = \cos y + y e^x \quad , \quad f_y = \frac{\partial f}{\partial y} = -x\sin y + e^x$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = y e^x \quad , \quad f_{yy} = \frac{\partial^2 f}{\partial y^2} = -x\cos y$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = -\sin y + e^x \quad , \quad f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = -\sin y + e^x$$

We see that $f_{xy} = f_{yx}$ or $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

Partial derivatives of higher order

Consider the 3rd order partial derivatives of the function $z = f(x,y)$

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial f_{xx}}{\partial x} = f_{xxx} \quad , \text{ similarly } \frac{\partial^3 f}{\partial y^3} = f_{yyy}.$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial f_{yx}}{\partial x} = f_{yxx}$$

$$\frac{\partial^3 f}{\partial y \partial x^2} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial f_{xx}}{\partial y} = f_{xxy}$$

We can show that $f_{yxx} = f_{xxy}$ as follows:

$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) = \frac{\partial}{\partial x}\left(\frac{\partial f_x}{\partial y}\right)$$

$$= \frac{\partial^2 f_x}{\partial x \partial y} = \frac{\partial^2 f_x}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f_x}{\partial x}\right) = \frac{\partial f_{xx}}{\partial y} =$$

$$= f_{xxy}$$

$$= \frac{\partial^3 f}{\partial y \partial x^2}$$

In general,

Ex.(17): Back to Ex.(16)

$$\frac{\partial^3 f}{\partial x^3} = y e^x \quad , \quad \frac{\partial^3 f}{\partial y^3} = +x \sin y$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial}{\partial x}\left(-\sin y + e^x\right) = e^x .$$

$$\frac{\partial^3 f}{\partial y \partial x^2} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial}{\partial y}\left(y e^x\right) = e^x .$$

we see that $\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial y \partial x^2}$.

In general $\frac{\partial^{m+n} f}{\partial x^m \partial y^n}$ denote the result of partial derivative of the function $f(x,y)$ with respect to x (m times) and with respect to y (n times)

Ex.(18): If $f(x,y) = x \sin y + e^y$. Find $\frac{\partial^5 f}{\partial x^2 \partial y^3}$

solu.

$$\frac{\partial f}{\partial x} = \sin y \Rightarrow \frac{\partial^2 f}{\partial x^2} = 0 \Rightarrow \frac{\partial^3 f}{\partial y \partial x^2} = 0 \Rightarrow \frac{\partial^4 f}{\partial y^2 \partial x^2} = 0$$

$$\Rightarrow \frac{\partial^5 f}{\partial y^3 \partial x^2} = 0$$

OR $\frac{\partial f}{\partial y} = x \cos y + e^y \Rightarrow \frac{\partial^2 f}{\partial y^2} = -x \sin y + e^y$

$$\Rightarrow \frac{\partial^3 f}{\partial y^3} = -x \cos y + e^y \Rightarrow \frac{\partial^4 f}{\partial x \partial y^3} = -\cos y$$

$$\Rightarrow \frac{\partial^5 f}{\partial x^2 \partial y^3} = 0 .$$

Ex.(19) If $z = x \ln(xy)$. Find $\frac{\partial^3 z}{\partial y^2 \partial x}$ and $\frac{\partial^4 z}{\partial x^2 \partial y^2}$

solu. $\frac{\partial f}{\partial y} = x \frac{x}{xy} = \frac{x}{y} \Rightarrow \frac{\partial^2 f}{\partial y^2} = -\frac{x}{y^2} \Rightarrow \frac{\partial^3 f}{\partial x \partial y^2} = -\frac{1}{y^2} = \frac{\partial^3 f}{\partial y^2 \partial x}$

$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = 0 .$$

Chain Rule for Higher Partial Derivatives of Functions with Many Variables :

EX.(20): If $w = f(x,y)$ where $x = e^t$, $y = 2t-1$. Find $\frac{d^2w}{dt^2}$ in terms of f_x , f_y , f_{xx} , f_{yy} , f_{xy} .

$$\frac{dw}{dt} = \frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$= f_x \cdot e^t + f_y \, 2$$

$$= e^t f_x + 2 f_y .$$

$$\frac{d^2w}{dt^2} = \frac{d}{dt}\left(\frac{dw}{dt}\right)$$

$$= e^t \frac{df_x}{dt} + e^t f_x + 2\frac{df_y}{dt}$$

$$= e^t f_x + e^t\left(\frac{\partial f_x}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f_x}{\partial y}\cdot\frac{dy}{dt}\right) + 2\left(\frac{\partial f_y}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f_y}{\partial y}\cdot\frac{dy}{dt}\right)$$

$$= e^t f_x + e^t\left(f_{xx}\, e^t + f_{xy}\cdot 2\right) + 2\left(f_{yx}\, e^t + f_{yy}\cdot 2\right).$$

$$= e^t f_x + e^{2t} f_{xx} + 2e^t f_{xy} + 2e^t f_{yx} + 4 f_{yy}$$

$$= e^t f_x + e^{2t} f_{xx} + 4e^t f_{xy} + 4 f_{yy}$$

where $f_{xy} = f_{yx}$.

EX.(21): If $z = f(u,v)$ where $u = x^2 - y^2$, $v = 2xy$. Find $z_{xy} = \frac{\partial^2 z}{\partial y \partial x}$ in terms ① $x, y, f_u, f_v, f_{uu}, f_{vv}, f_{uv}$
② $u, v, f_u, f_v, f_{uu}, f_{vv}, f_{uv}$

solu.

$$z_x = \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$
$$= 2x\, f_u + 2y\, f_v$$

$$z_{xy} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(2x f_u + 2y f_v\right)$$
$$= 2x\frac{\partial f_u}{\partial y} + 2y\frac{\partial f_v}{\partial y} + 2 f_v$$

$$z_{xy} = 2x\left(\frac{\partial f_u}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f_u}{\partial v}\cdot\frac{\partial v}{\partial y}\right) + 2y\left(\frac{\partial f_v}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f_v}{\partial v}\cdot\frac{\partial v}{\partial y}\right) + 2 f_v$$
$$= 2x\left(-2y\, f_{uu} + 2x\, f_{uv}\right) + 2y\left(-2y\, f_{vu} + 2x\, f_{vv}\right) + 2 f_v$$
$$= -4xy\, f_{uu} + 4x^2 f_{uv} - 4y^2 f_{vu} + 4xy\, f_{vv} + 2 f_v$$
$$= -4xy\, f_{uu} + 4(x^2 - y^2) f_{uv} + 4xy\, f_{vv} + 2 f_v \quad \boxed{\text{where } f_{uv} = f_{vu}}$$
$$= 4xy\left(f_{vv} - f_{uu}\right) + 4(x^2 - y^2) f_{uv} + 2 f_v$$
$$\therefore\ z_{xy} = 2v\left(f_{vv} - f_{uu}\right) + 4u\, f_{uv} + 2 f_v$$

Tangent Plane & Normal Line

هنا لدينا سطح surface معادلته $z=f(x,y)$ وعليه نقطة
هي $P_0(x_0,y_0,z_0)$ ، المستوي المماس للسطح عند النقطة
P_0 يسمى بـ Tangent plane .
المستقيم العمودي على المستوي المماس يسمى بـ Normal line .
والمطلوب هنا هو ايجاد معادلة المستوي المماس والمستقيم العمودي .

<u>Theorem</u> Let $P_0(x_0, y_0, z_0)$ be any point on the surface $z=f(x,y)$. If $f(x,y)$ is differentiable at (x_0,y_0), then the surface has a tangent plane at P_0 and this plane has the equation

$$f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)-(z-z_0)=0$$

<u>or</u>

$$\left.\frac{\partial f}{\partial x}\right|_{x_0,y_0}\cdot(x-x_0)+\left.\frac{\partial f}{\partial y}\right|_{x_0,y_0}\cdot(y-y_0)-(z-z_0)=0 .$$

The parametric equations of the normal line at P_0 are

$x = x_0 + f_x(x_0, y_0)t$ or $x = x_0 + \frac{\partial f}{\partial x}\Big|_{x_0, y_0} \cdot t$

$y = y_0 + f_y(x_0, y_0)t$ or $y = y_0 + \frac{\partial f}{\partial y}\Big|_{x_0, y_0} \cdot t$

$z = z_0 - t$ $\quad z = z_0 - t$

<u>Example:</u> Find equations of the tangent plane and normal line to the surface $z = f(x,y) = x^2 y$ at the point $(2,1,4)$.

<u>Solu.</u> $f(x,y) = x^2 y$

$f_x(x,y) = 2xy$ and $f_y(x,y) = x^2$

$f_x(2,1) = 4$ and $f_y(2,1) = 4$

The tangent plane has the equation

$4(x-2) + 4(y-1) - (z-4) = 0$

and the normal line has equations

$x = 2 + 4t$
$y = 1 + 4t$
$z = 4 - t$

<u>Example:</u> Find equations of the tangent plane and normal line to the surface $x^2 + y^2 + z^2 = 25$ at the point $P(-3, 0, 4)$.

<u>Solu.</u> We shall differentiate the surface equation ~~implicitly~~ implicitly.

$x^2 + y^2 + z^2 = 25$ and $2y + 2z\frac{\partial z}{\partial y} = 0$

$2x + 0 + 2z\frac{\partial z}{\partial x} = 0$ $\quad \frac{\partial z}{\partial y} = \frac{-2y}{2z} = \frac{-y}{z}$

$\frac{\partial z}{\partial x} = \frac{-2x}{2z} = -\frac{x}{z}$

$$\left.\frac{\partial z}{\partial x}\right|_{-3,0,4} = -\frac{-3}{4} = \frac{3}{4} \qquad \left.\frac{\partial z}{\partial y}\right|_{-3,0,4} = 0$$

The tangent plane has the equation

$$\frac{3}{4}(x+3) + 0 - (z-4) = 0$$

simplifing

$$3x - 4z = -25 \quad .$$

Equations of normal Line:

$$x = -3 + \frac{3}{4}t$$
$$y = 0 + 0 = 0$$
$$z = 4 - t \quad .$$

Directional Derivative and Gradient

Directional derivative: المشتقة الاتجاهية

موضوعنا هذا يخص إيجاد ميل المماس للسطح $f(x,y)$ عند النقطة (x,y) في اتجاهات أخرى غير التي توازي محاور x,y .

فلو كان لدينا سطح $z=f(x,y)$ (مرسوم في 3-D) ومستقيم l مرسوم في (2-D) على مستوى xy يميل المماس للسطح عند النقطة (x_0,y_0) بإتجاه المستقيم l يسمى بـ:

directional derivative of f at (x_0,y_0) in the direction of f of $\vec{u}$

أي المشتقة الإتجاهية للدالة f عند (x_0,y_0) بموازاة متجه الوحدة $\vec{u}$ حيث أن $\vec{u}$ هو unit vector المطلوب لإيجاد المشتقة الإتجاهية بموازاته.

<u>Defn.</u> If $f(x,y)$ is differentiable at (x_0,y_0) and if $\vec{u}=u_1 i+u_2 j$ is a unit vector, then the directional derivative of f at (x_0,y_0) in the direction of $\vec{u}$ is defined as

$$D_u f(x_0,y_0) = f_x(x_0,y_0)\,u_1 + f_y(x_0,y_0)\,u_2$$
$$= \left.\frac{\partial f}{\partial x}\right|_{x_0,y_0} u_1 + \left.\frac{\partial f}{\partial y}\right|_{x_0,y_0} u_2 .$$

<u>Ex.</u> Find the directional derivative of $f(x,y)=3x^2y$ at the point $(1,2)$ in the direction of $\vec{a}=3i+4j$

<u>solu.</u>

$$\vec{u}=\frac{\vec{a}}{\|\vec{a}\|}=\frac{3i+4j}{\sqrt{3^2+4^2}}=\overset{u_1}{\frac{3}{5}}\,i+\overset{u_2}{\frac{4}{5}}\,j .$$

$$\frac{\partial f}{\partial x}=6xy \qquad \left.\frac{\partial f}{\partial x}\right|_{1,2}=12$$

$$\frac{\partial f}{\partial y}=3x^2 \qquad \left.\frac{\partial f}{\partial y}\right|_{1,2}=3$$

$$D_u f(1,2)=12\left(\frac{3}{5}\right)+3\left(\frac{4}{5}\right)=\frac{48}{5} .$$

EX. Find the directional derivative of e^{xy} at $(-2,0)$ in the direction of an angle that makes $\frac{\pi}{3}$ with the positive x-axis.

Solu.

$$f(x,y) = e^{xy} \Rightarrow f_x = y e^{xy} \Rightarrow f_x\Big|_{-2,0} = 0$$

$$f_y = x e^{xy} \Rightarrow f_y\Big|_{-2,0} = -2$$

$$\vec{u} = \cos\frac{\pi}{3}\, i + \sin\frac{\pi}{3}\, j$$
$$= \frac{1}{2} i + \frac{\sqrt{3}}{2} j$$

$$D_u f(-2,0) = 0 + (-2)\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{3}.$$

Defn. If f is differentiable at (x_0, y_0, z_0) and if $\vec{u} = u_1 i + u_2 j + u_3 k$ is a unit vector, then the directional derivative of f at (x_0, y_0, z_0) in the direction of $\vec{u}$ is defined by

$$D_u f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0) \cdot u_1 + f_y(x_0, y_0, z_0) \cdot u_2 + f_z(x_0, y_0, z_0) \cdot u_3$$

$$= \frac{\partial f}{\partial x}\Big|_{x_0, y_0, z_0} \cdot u_1 + \frac{\partial f}{\partial y}\Big|_{x_0, y_0, z_0} \cdot u_2 + \frac{\partial f}{\partial z}\Big|_{x_0, y_0, z_0} \cdot u_3$$

EX. Find the directional derivative of $f(x,y,z) = x^2y - yz^3 + z$ at point $p(1,-2,0)$ in the direction of the vector $\vec{a} = 2i + j - 2k$.

Solu.

$$\vec{u} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{2i + j - 2k}{\sqrt{4+1+4}} = \frac{2}{3}i + \frac{1}{3}j - \frac{2}{3}k.$$

$$\frac{\partial f}{\partial x} = 2yx \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{1,-2,0} = -4$$

$$\frac{\partial f}{\partial y} = x^2 - z^3 \Rightarrow \left.\frac{\partial f}{\partial y}\right|_{1,-2,0} = 1$$

$$\frac{\partial f}{\partial z} = -3yz^2 + 1 \Rightarrow \left.\frac{\partial f}{\partial z}\right|_{1,-2,0} = 1$$

$$D_u f(1,-2,0) = -4\left(\frac{2}{3}\right) + 1\left(\frac{1}{3}\right) + 1\left(\frac{-2}{3}\right) = -3.$$

The Gradient : The directional derivative formula

$D_u f(x,y) = f_x(x,y)u_1 + f_y(x,y)u_2$. Can be expressed in the form of a dot product by writing

$$D_u f(x,y) = \left(f_x(x,y)\,i + f_y(x,y)\,j\right) \cdot (u_1 i + u_2 j).$$

The second vector in the dot product is the unit vector u. However, the first vector is something new; it is called the (gradient) of f and is denoted by the symbol ∇f or $\nabla f(x,y)$.

Defn. If f is a function of x and y, then the gradient of f is defined by

$$\nabla f(x,y) = f_x(x,y)\,i + f_y(x,y)\,j \quad \underline{\underline{\text{or}}} \quad \nabla f(x,y) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$$

With the gradient notation, the directional derivative can be written in the following compact form

$$D_u f(x,y) = \nabla f(x,y) \cdot \vec{u}$$

In other words, the dot product of the gradient of f with a unit vector $\vec{u}$ produces the directional derivative of f in the direction of $\vec{u}$.

Ex. Find the gradient of $f(x,y) = 3x^2y$ at the point $(1,2)$ and use it to calculate the directional derivative of f at $(1,-2)$ in the direction of the vector $\vec{a} = 3i + 4j$.

Solu. We have $\nabla f(x,y) = f_x(x,y)\,i + f_y(x,y)\,j$

$$\nabla f(x,y) = 6xy\,i + 3x^2 j$$

So that the gradient of f at $(1,2)$ is

$$\nabla f(1,2) = 12i + 3j$$

The unit vector in the direction of $\vec{a}$ is

$$\vec{u} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{3i+4j}{\sqrt{9+16}} = \frac{3}{5}i + \frac{4}{5}j$$

$$\therefore D_u f(1,2) = \nabla f(1,2) \cdot \vec{u} = (12i + 3j)\left(\frac{3}{5}i + \frac{4}{5}j\right)$$

$$= \frac{48}{5}$$

<u>Example:</u> Given that $D_u f(1,2) = -5$ if $\vec{u} = \frac{3}{5}i - \frac{4}{5}j$ and $D_v f(1,2) = 10$ if $\vec{v} = \frac{4}{5}i + \frac{3}{5}j$, find

(a) $f_x(1,2)$, (b) $f_y(1,2)$, (c) The directional derivative of f at $(1,2)$ in the direction of the <u>origin</u>

<u>Solu.</u> (a) and (b) note that $\vec{u}$ and $\vec{v}$ are unit vectors

$$\left.\begin{aligned} D_u f(1,2) &= \nabla f(1,2)\cdot\vec{u} \\ D_v f(1,2) &= \nabla f(1,2)\cdot\vec{v} \end{aligned}\right\}$$

$$\left.\begin{aligned} -5 &= \left(f_x(1,2)\,i + f_y(1,2)\,j\right)\left(\tfrac{3}{5}i - \tfrac{4}{5}j\right) \\ 10 &= \left(f_x(1,2)\,i + f_y(1,2)\,j\right)\left(\tfrac{4}{5}i + \tfrac{3}{5}j\right) \end{aligned}\right\}$$

$$\left.\begin{aligned} -5 &= f_x(1,2)\,\tfrac{3}{5} + f_y(1,2)\left(-\tfrac{4}{5}\right) \\ 10 &= f_x(1,2)\,\tfrac{4}{5} + f_y(1,2)\left(\tfrac{3}{5}\right) \end{aligned}\right\}$$

Solving for $f_x(1,2)$ and $f_y(1,2)$, we get:

$$f_x(1,2) = 5 \quad , \quad f_y\Big|_{(1,2)} = 10$$

(c) The vector $\vec{a}$ is $\vec{a} = -i - 2j$

$$\frac{\vec{a}}{\|\vec{a}\|} = \frac{-i-2j}{\sqrt{5}} \Longrightarrow \text{unit vector of } \vec{a}$$

$$D_u f(1,2) = \left(f_x(1,2)\,i + f_y(1,2)\,j\right)\left(\frac{-i-2j}{\sqrt{5}}\right)$$

$$= \left(5i + 10j\right)\left(\frac{-i-2j}{\sqrt{5}}\right)$$

$$= \frac{-5}{\sqrt{5}} - \frac{20}{5} = -5\sqrt{5}.$$