Steps for Solving other Form of Linear First Order Equation

There is another form of differential equation that can be written in the form

$$\frac{dx}{dy} + P(y)x = Q(y)$$

where P and Q are functions of y. The solution is found as follows:

i. Put it in standard form and identify the functions P and Q.

ii. Find an anti-derivative of $P(y)$.

iii. Find the integrating factor $\rho(y) = e^{\int P(y)dy}$.

iv. Find x using the following equation

$$x = \frac{1}{\rho(y)} \int \rho(y)Q(y)dy$$

Example

Solve the equation $e^{2y}dx + 2(xe^{2y} - y)dy = 0$.

Solution

Dividing the differential equation by $e^{2y}dy$ to get

$$\frac{dx}{dy} + 2x - 2ye^{-2y} = 0$$

$$\frac{dx}{dy} + 2x = 2ye^{-2y} \qquad \Rightarrow \qquad P(y) = 2, \quad Q(y) = 2ye^{-2y}$$

$$\int P(y)dy = \int 2dy = 2y, \qquad \rho(y) = e^{\int P(y)dy} = e^{2y}$$

$$x = \frac{1}{e^{2y}} \int (e^{2y})(2ye^{-2y})dy + C \quad \Rightarrow \quad e^{2y}x = 2\int ydy + C$$

$$e^{2y}x = 2\frac{y^2}{2} + C \qquad \Rightarrow \qquad e^{2y}x = y^2 + C$$

Reducible to Linear

- The general form

$$\frac{dy}{dx} + P(x)y = Q(x)f(y)$$

where the function f is y to any power n.

- Also, it may be in the following form

$$\frac{dy}{dx} + P(x)g(y) = Q(x)h(y)$$

where the function g and h are functions of y.

Example

Solve the equation $\frac{dy}{dx} + \frac{y}{x} = \ln(x)y^2$

Solution

Dividing the two sides of the equation by y^2

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \ln(x)$$

Let $z = \frac{1}{y} \quad \Rightarrow \quad \frac{dz}{dx} = -\frac{1}{y^2}\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = -y^2\frac{dz}{dx}$

$$-\frac{dz}{dx} + \frac{1}{x}z = \ln(x)$$

$$\frac{dz}{dx} - \frac{1}{x}z = -\ln(x) \qquad \Rightarrow \quad P = \frac{-1}{x}, \quad Q = -\ln(x)$$

$$\int P(x)dx = \int \frac{-1}{x}dx = -\ln(x)$$

$$\rho(x) = e^{\int P(x)dx} = e^{-\ln(x)} = e^{\ln(x)^{-1}} = e^{\ln\left(\frac{1}{x}\right)} = \frac{1}{x}$$

$$\rho(x)z = \int \rho(x)Q(x)dx + C$$

$$\frac{1}{x}z = -\int \frac{1}{x}\ln(x)dx + C$$

$$\frac{1}{x} \times \frac{1}{y} = -\frac{(\ln(x))^2}{2} + C \quad \Rightarrow \quad \frac{1}{xy} = -\frac{(\ln(x))^2}{2} + C$$

Example

Solve the equation $\frac{dy}{dx} + x\sin(2y) = x\cos^2(y)$

Solution

Dividing the two sides of the equation by $\cos^2(y)$

$$\frac{1}{\cos^2(y)}\frac{dy}{dx} + x\frac{\sin(2y)}{\cos^2(y)} = x \quad \Rightarrow \quad \sec^2(y)\frac{dy}{dx} + x\frac{2\sin(y)\cos(y)}{\cos^2(y)} = x$$

$$\sec^2(y)\frac{dy}{dx} + x\frac{2\sin(y)}{\cos(y)} = x \quad \Rightarrow \quad \sec^2(y)\frac{dy}{dx} + 2x\tan(y) = x$$

Let $z = \tan(y) \quad \Rightarrow \quad \frac{dz}{dx} = \sec^2(y)\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec^2(y)}\frac{dz}{dx}$

$$\frac{dz}{dx} + 2xz = x \quad \Rightarrow \quad P = 2x, \quad Q = x$$

$$\int P(x)dx = \int 2xdx = x^2 \quad \Rightarrow \quad \rho(x) = e^{\int P(x)dx} = e^{x^2}$$

$$\rho(x)z = \int \rho(x)Q(x)dx + C$$

$$e^{x^2}z = \int e^{x^2}(x)dx + C \quad \Rightarrow \quad e^{x^2}\tan(y) = \frac{e^{x^2}}{2} + C$$

Another Form of Reducible to Linear

- The general form may be as follows

$$\frac{dx}{dy} + P(y)x = Q(y)f(x)$$

where the function f is x to any power n.

- Also, it may be in the following form

$$\frac{dx}{dy} + P(y)g(x) = Q(y)h(x)$$

where the function g and h are functions of x.

Example

Solve the equation $\cos(y)dx = x(\sin(y) - x)dy$

Solution

Dividing the two sides of the equation by $\cos(y)dy$

$$\frac{dx}{dy} = \frac{\sin(y)}{\cos(y)}x - \frac{x^2}{\cos(y)} \quad \Rightarrow \quad \frac{dx}{dy} - x\tan(y) = -x^2\sec(y)$$

Dividing by x^2, we get

$$\frac{1}{x^2}\frac{dx}{dy} - \frac{1}{x}\tan(y) = -\sec(y)$$

Let $z = \frac{1}{x} \quad \Rightarrow \quad \frac{dz}{dy} = \frac{-1}{x^2}\frac{dx}{dy} \quad \Rightarrow \quad \frac{dx}{dy} = -x^2\frac{dz}{dy}$

$$-\frac{dz}{dy} - z\tan(y) = -\sec(y)$$

$$\frac{dz}{dy} + z\tan(y) = \sec(y). \quad \Rightarrow \quad P = \tan(y), \quad Q = \sec(y)$$

$$\int P(y)dy = \int \tan(y)dy = \int \frac{\sin(y)}{\cos(y)}dy = -\ln(\cos(y))$$

$$\rho(y) = e^{\int P(y)dy} = e^{-\ln(\cos(y))} = e^{\ln(\cos(y))^{-1}} = e^{\ln\left(\frac{1}{\cos(y)}\right)} = \sec(y)$$

$$\rho(y)z = \int \rho(y)Q(y)dy + C$$

$$\sec(y) \times \frac{1}{x} = \int \sec(y)\sec(y)dy + C$$

$$\frac{\sec(y)}{x} = \int \sec^2(y)dy + C \quad \Rightarrow \quad \frac{\sec(y)}{x} = \tan(y) + C$$

Exercises

Find the solution of the following Differential Equations

1) $y' - y = 3$
2) $y' + 2xy = 0$
3) $y' + 2y = 6e^x$
4) $y' - 4y = 2x - 4x^2$
5) $y' + y = \sin(x)$
6) $y' + 2y = \cos(x)$
7) $y' + ky = e^{-kx}$
8) $y' = (y-1)\cot(x)$
9) $xy' - 2y = x^3e^x$
10) $x^2y' + 2xy = \sinh(3x)$
11) $y' - y = e^x,\ y(1) = 0$
12) $y' + y = (x+1)^2,\ y(0) = 0$
13) $y' + xy = xy^{-1}$
14) $y' + x^{-1}y = x^{-1}y^2$
15) $2xy' = 10x^3y^5 + y$
16) $y' + y = xy^{-1}$

Exact Differential Equations

Example

If $f(x,y)=C$ and $f(x,y)=\sin(xy)$ then

$$\frac{df}{dx}=y\cos(xy)+x\cos(xy)\frac{dy}{dx}=0, \text{ or}$$

$$df=y\cos(xy)dx+x\cos(xy)dy=0$$

i.e., $y\cos(xy)dx+x\cos(xy)dy=0$

From the above equation, we see that $M(x,y)=y\cos(xy)=\dfrac{\partial f}{\partial x}$, and $N(x,y)=x\cos(xy)=\dfrac{\partial f}{\partial y}$. The solution of this differential equation is $f(x,y)=C$.

Exact Differential Equation Test

A differential equation $M(x,y)dx+N(x,y)dy=0$ is said to be *exact* if for some function $f(x,y)$

$$M(x,y)dx+N(x,y)dy=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=df$$

is exact if and only if $$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Example

- The equation $(x^2+y^2)dx+(2xy+\cos(y))dy=0$ is exact because the partial derivatives

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}(x^2+y^2)=2y, \qquad \frac{\partial N}{\partial x}=\frac{\partial}{\partial x}(2xy+\cos(y))=2y$$

are equal.

- The equation $(x+3y)dx+(x^2+\cos(y))dy=0$ is not exact because the partial derivatives

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}(x+3y)=3, \qquad \frac{\partial N}{\partial x}=\frac{\partial}{\partial x}(x^2+\cos(y))=2x$$

are not equal.

Steps for Solving an Exact Differential Equation

i. Match the equation to the form $M(x,y)dx+N(x,y)dy=0$ to identify M and N.

ii. Integrate M (or N) with respect to x (or y), writing the constant of integration as $g(y)$ (or $g(x)$).

iii. Differentiate with respect to y (or x) and set the result equal to N (or M) to find $g'(y)$ (or $g'(x)$).

iv. Integrate to find $g(y)$ (or $g(x)$).

v. Write the solution of the exact equation as $f(x,y)=C$.

Example

Solve the differential equation

$$(x^2+y^2)dx+(2xy+\cos(y))dy=0.$$

Solution

Step 1: ***Match the equation to the form*** $M(x,y)dx+N(x,y)dy=0$ ***to identify*** M.

$$M(x,y)=x^2+y^2$$

Step 2: ***Integrate*** M ***with respect to*** x, ***writing the constant of integration as*** $g(y)$.

$$f(x,y) = \int M(x,y)dx = \int (x^2 + y^2)dx = \frac{x^3}{3} + xy^2 + g(y)$$

Step 3: ***Differentiate with respect to*** y ***and set the result equal to*** N ***to find*** $g'(y)$.

$$\frac{\partial}{\partial y}\left(\frac{x^3}{3} + xy^2 + g(y)\right) = 2xy + g'(y)$$

$$2xy + g'(y) = 2xy + \cos(y) \quad \Rightarrow \quad g'(y) = \cos(y)$$

Step 4: ***Integrate to find*** $g(y)$.

$$\int g'(y)dy = \int \cos(y)dy = \sin(y)$$

Step 5: ***Write the solution of the exact equation as*** $f(x,y) = C$.

$$\frac{x^3}{3} + xy^2 + \sin(y) = C$$

Another Solution

Step 1: ***Match the equation to the form*** $M(x,y)dx + N(x,y)dy = 0$ ***to identify*** N.

$$N(x,y) = 2xy + \cos(y)$$

Step 2: ***Integrate*** N ***with respect to*** y, ***writing the constant of integration as*** $g(x)$.

$$f(x,y) = \int N(x,y)dy = \int (2xy + \cos(y))dy = xy^2 + \sin(y) + g(x)$$

Step 3: ***Differentiate with respect to*** x ***and set the result equal to*** M ***to find*** $g'(x)$.

$$\frac{\partial}{\partial x}\left(xy^2 + \sin(y) + g(x)\right) = y^2 + g'(x)$$

$$y^2 + g'(x) = x^2 + y^2 \quad \Rightarrow \quad g'(x) = x^2$$

Step 4: ***Integrate to find*** $g(x)$.

$$\int g'(x)dx = \int x^2 dx = \frac{x^3}{3}$$

Step 5: ***Write the solution of the exact equation as*** $f(x,y) = C$.

$$\frac{x^3}{3} + xy^2 + \sin(y) = C$$

Reducible to Exact

A differential equation $M(x,y)dx + N(x,y)dy = 0$ which is not exact can be made exact by multiplying both sides by a suitable integrating factor ρ. In other words, the equation

$$\rho M(x,y)dx + \rho N(x,y)dy = 0$$

is an exact equation for an appropriate choice of ρ.

Method to Find the Integrating Factor

- If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$ or *Constant* then $\rho(x) = e^{\int f(x)dx}$.
- If $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = f(y)$ or *Constant* then $\rho(y) = e^{\int f(y)dy}$.

Example

Solve the equation $2ydx + xdy = 0$

Solution

$$M(x,y) = 2y \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 2$$

$$N(x,y) = x \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 1$$

This equation is not exact

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2-1}{x} = \frac{1}{x} = f(x)$$

$$\int f(x)dx = \int \frac{1}{x} dx = \ln(x)$$

$$\rho(x) = e^{\int f(x)dx} = e^{\ln(x)} = x$$

Multiplying both sides of the equation by the integrating factor $\rho(x) = x$, we get

$$x(2ydx + xdy) = 0 \quad \Rightarrow \quad 2xydx + x^2dy = 0$$

which is exact because $\frac{\partial M}{\partial y} = 2x$ and $\frac{\partial N}{\partial x} = 2x$, and the solution is

$$f(x,y) = \int 2xydx = x^2y + g(y)$$

$$\frac{\partial}{\partial y}\left(x^2y + g(y)\right) = x^2 + g'(y)$$

$$x^2 + g'(y) = x^2 \quad \Rightarrow \quad g'(y) = 0$$

$$g(y) = \int g'(y)dy = C \quad \Rightarrow \quad x^2y = C$$

Exercises

Find the solution of the following Differential Equations

1) $ydx + xdy = 0$

2) $(xdy - ydx)/x^2 = 0$

3) $(2x + e^y)dx + xe^y dy = 0$

4) $2x\ln(y)dx + y^{-1}x^2 dy = 0$

5) $\sinh(x)\cos(y)dx = \cosh(x)\sin(y)dy$

6) $3re^{3\theta}d\theta + e^{3\theta}dr = 0$

7) $(1 + x^2)dy + 2xydx = 0$

8) $xdy - 4ydx = 0$

9) $ydx + x(1 + y)dy = 0$

10) $(2ydx + dy)e^{2x} = 0$

11) $(3y\cos(3x)dx - \sin(3x)dy)/y^2 = 0$

12) $\sin(\beta y)dx = -\beta\cos(\beta y)dy$

13) $xdy - ydx = 0$

14) $2\cos(\pi y)dx = \pi\sin(\pi y)dy$

15) $y\cos(x)dx + 3\sin(x)dy = 0$

16) $3ydx + 2xdy = 0$

17) $dx + (y/x)^2 dy = 0$

18) $2dx - e^{y-x}dy = 0$

19) $y\cos(x)dx + 2\sin(x)dy = 0$

20) $(y + 1)dx - (x + 1)dy = 0$

21) $2ydx + xdy = 0$

22) $\sin(y)dx + \cos(y)dy = 0$

23) $2dx + \sec(x)\cos(y) = 0,\ y(0) = 0$

24) $2x^2dx - 3xy^2dy = 0, y(1) = 0$

25) $2\sin(y)dx + \cos(y)dy = 0,$

$y(0) = \pi/2$

26) $(2y + xy)dx + 2xdy = 0,$

$y(3) = \sqrt{2}$