Civil Engineering
1st. Year

Infinite Sequences & Series

Sequences of numbers

examples

Sequence of positive integers $1, 2, 3, 4, \ldots$
" " even " $2, 4, 6, 8, \ldots$
" " prime " $1, 3, 5, 7, \ldots$
" " squares $1, 4, 9, 16, \ldots$

$\left.\right\}$ do not approach finite limits

" " reciprocals $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$ approchas to zero as $n \longrightarrow \infty$

Def. A sequence of numbers is a function whose domain is the set of positive integers (natural numbers).

Sequences are defined by rules, such as,

$a(n) = n-1$, $a(n) = 1 - \frac{1}{n}$, $a(n) = \frac{\ln n}{n^2}$ $\quad n = 1, 2, 3, \ldots$

$a(n)$ or simply a_n is called the n-th term of the sequence

ex let $a_n = \frac{n-1}{n}$ then $a_1 = 0$, $a_2 = \frac{1}{2}$, $a_3 = \frac{2}{3}$, ---

ex

The Terms	The n-th term	The Sequence
$0, 1, 2, 3, \ldots$	$n-1$	$\{n-1\}$
$1, \frac{1}{2}, \frac{1}{3}, \ldots$	$\frac{1}{n}$	$\{\frac{1}{n}\}$
$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots$	$(-1)^{n+1}(\frac{1}{n})$	$\{(-1)^{n+1}\frac{1}{n}\}$
$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$	$\frac{n-1}{n}$	$\{\frac{n-1}{n}\}$
$0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \ldots$	$(-1)^{n+1}(\frac{n-1}{n})$	$\{(-1)^{n+1}(\frac{n-1}{n})\}$
$3, 3, 3, 3, \ldots$	3	$\{3\}$

Infinite Series

Given a sequence of numbers $\{a_n\}$, an expression of the form $a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n$ is called an infinite series. The number a_n is called the n-th term of the series.

Now let $s_1 = a_1$
$s_2 = a_1 + a_2$
$s_3 = a_1 + a_2 + a_3$
$\vdots$
$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k \implies \{S_n\}$

The new sequence is called the sequence of partial sums of the series

If $\{S_n\}$ convergers to a limit L, then the series converges and its sum is L.

If $\{S_n\}$ diverges, then the series diverges.

ex $0.33333\cdots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots$

$S_1 = \frac{3}{10}$, $S_2 = \frac{3}{10} + \frac{3}{100}$, $S_3 = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} \cdots$

$S_n = \frac{3}{10} + \frac{3}{(10)^2} + \frac{3}{(10)^3} + \cdots + \frac{3}{(10)^n}$ --- ①

Now, by multiplying ① by $\frac{1}{10}$ we have:

$\frac{1}{10} S_n = \frac{3}{(10)^2} + \frac{3}{(10)^3} + \frac{3}{(10)^4} + \cdots + \frac{3}{(10)^{n+1}}$ --- ②

subtracting ② from ① we get:

$$S_n - \frac{1}{10} S_n = \frac{3}{10} - \frac{3}{10^{n+1}} = \frac{3}{10}\left(1 - \frac{1}{10^n}\right)$$

$$\frac{9}{10} S_n = \frac{3}{10}\left(1 - \frac{1}{10^n}\right) \implies \boxed{S_n = \frac{1}{3}\left(1 - \frac{1}{10^n}\right)}$$

Now, as $n \to \infty$, $S_n \to \frac{1}{3}$ since $\frac{1}{10^n} \to 0$

$\therefore \lim_{n \to \infty} S_n = \frac{1}{3}$

Hence we say that the sum of the infinite series

$\frac{3}{10} + \frac{3}{10^2} + \cdots + \frac{3}{10^n} + \cdots$ is $\frac{1}{3}$

or $\sum_{n=1}^{\infty} \frac{3}{10^n} = \frac{1}{3}$

Geometric Series

$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ is called Geometric Series

$S_n = \dfrac{a(1-r^n)}{1-r}$, $r \neq 1$, $S_n = na$, $r = 1$

if $|r| < 1$ then $\lim\limits_{n\to\infty} = \lim\limits_{n\to\infty} \dfrac{a(1-r^n)}{1-r} = \dfrac{a}{1-r} \Rightarrow \sum\limits_{n=1}^{\infty} ar^{n-1}$ converge

if $|r| \geq 1$ ~~then~~ $\lim\limits_{n\to\infty} S_n = \infty \Rightarrow \sum\limits_{n=1}^{\infty} ar^{n-1}$ diverges

ex $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum\limits_{n=1}^{\infty} \dfrac{1}{2^{n-1}}$ is gemotric series

$\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \dfrac{1\,(1-(\frac{1}{2})^n)}{1-\frac{1}{2}} = \dfrac{1}{1/2} = 2$

$\therefore \sum\limits_{n=1}^{\infty} \dfrac{1}{2^{n-1}}$ converges to 2

Notes

① if $\sum\limits_{n=1}^{\infty} u_n$ converges $\Longrightarrow \lim\limits_{n\to\infty} u_n = 0$

② if $\lim\limits_{n\to\infty} u_n = 0 \Rightarrow \sum\limits_{n=1}^{\infty} u_n$ either converges or diverges

③ if $\lim\limits_{n\to\infty} u_n \neq 0 \Longrightarrow \sum\limits_{n=1}^{\infty} u_n$ diverges.

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ex $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n}{n+1} + \cdots$ diverges since

$\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \dfrac{n}{n+1} = \lim\limits_{n\to\infty} \dfrac{1}{1+\frac{1}{n}} = 1 \neq 0$

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ex $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots = \sum\limits_{n=1}^{\infty} \dfrac{1}{n}$

$\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$

$\therefore \sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ either converges or diverges (diverges, later on)

since $1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{>1/2} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{>1/2} + \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right)}_{>1/2} + \cdots$

$\therefore \sum \frac{1}{n} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$ and this is not finite

Note: If $\sum_{n=1}^{\infty} U_n$ and $\sum_{n=1}^{\infty} V_n$ converge and c is any real number then $\sum_{n=1}^{\infty} cU_n$ and $\sum_{n=1}^{\infty}(U_n \mp V_n)$ converge too.

$$\sum_{n=1}^{\infty} cU_n = c\sum_{n=1}^{\infty} U_n$$

$$\sum_{n=1}^{\infty}(U_n \mp V_n) = \sum_{n=1}^{\infty} U_n \mp \sum_{n=1}^{\infty} V_n$$

Test of Convergence (of series of non-negative terms)

① **P-Test**

$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$ is called P-series

and converges if $p > 1$ and diverges if $p \leq 1$

ex $\sum_{n=1}^{\infty} \frac{1}{n}$ div. , $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ div.

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. , $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ conv.

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② **Comparison Test**: For two non-negative series $\sum_{n=1}^{\infty} U_n$, $\sum_{n=1}^{\infty} V_n$

ⓐ If $\sum_{n=1}^{\infty} V_n$ is known to be a convergence series, then $\sum_{n=1}^{\infty} U_n$ converges too, if $U_n \leq V_n$, $\forall n$.

ⓑ If $\sum_{n=1}^{\infty} V_n$ is known to be a divergence series, then $\sum_{n=1}^{\infty} U_n$ diverges too, if $U_n \geq V_n$, $\forall n$.

ex Test for convergence $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$

Sol $U_n = \frac{1}{n(n+1)}$

$\because (n+1) \geq n \Rightarrow n(n+1) \geq n^2 \Rightarrow \frac{1}{n(n+1)} \leq \frac{1}{n^2} \quad \forall n$

$\because \sum \frac{1}{n^2}$ converges by P-test

$\therefore \sum \frac{1}{n(n+1)}$ converges by Comparison test.

ex Test for convergence $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

sol. $\because \ln n < n \Rightarrow \frac{1}{\ln n} > \frac{1}{n}$, $\forall n > 1$

$\because \sum \frac{1}{n}$ diverges by P-Test

$\therefore \sum \frac{1}{\ln n}$ diverges by comparison test.

ex $\sum \frac{\ln n}{2n^3 - 1}$

sol. $\ln n < n$, $\forall n > 1$ and $2n^3 - 1 \geq n^3$, $\forall n > 1$

$\Rightarrow \frac{1}{2n^3-1} < \frac{1}{n^3}$

$\therefore \frac{\ln n}{2n^3 - 1} < \frac{n}{n^3} = \frac{1}{n^2}$

$\therefore \sum \frac{1}{n^2}$ converges by P-test.

$\therefore \sum \frac{\ln n}{2n^3-1}$ converges by comparison test.

ex $\sum \frac{1}{n!}$,

$\because n! \geq 2^{n-1}$, $\forall n$

$\frac{1}{n!} \leq \frac{1}{2^{n-1}}$

$\because \sum \frac{1}{2^{n-1}}$ converges because it is geometric series $|r| = \frac{1}{2} < 1$

$$\left(\begin{array}{l} n! = n(n-1)(n-2)\cdots 3.2.1 \\ 1! = 1 \geq 2^0 \\ 2! = 2.1 = 2 \geq 2^1 \\ 3! = 3.2.1 = 6 \geq 2^2 \\ 4! = 4.3.2.1 = 24 \geq 2^3 \\ 5! = 5.4.3.2.1 = 120 \geq 2^4 \\ \vdots \\ n! > 2^{n-1} \end{array}\right)$$

$\therefore \sum \frac{1}{n!}$ converges by comparison test

③ **Integral Test**: Suppose there is a decreasing continuous function $f(x)$, such that $f(n) = u_n$ is the n-th term of the +ve series $\sum_{n=1}^{\infty} u_n$. Then the series $\sum_{n=1}^{\infty} u_n$ and the integral $\int_1^{\infty} f(x)\,dx$ both converge or diverge.

ex $\sum_{n=1}^{\infty} \frac{1}{n+10}$

Sol $f(x) = \frac{1}{x+10}$, $f(x)$ is +ve cont., decreasing

$$\int_1^{\infty} f(x)\,dx = \lim_{a\to\infty} \int_1^{a} \frac{dx}{x+10} = \lim_{a\to\infty} \left[\ln(x+10)\right]_1^{a}$$

$$= \lim_{a\to\infty} \ln(a+10) - \ln 11 = \infty \quad \text{(diverges)}$$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n+10}$ diverges by integral test.

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ex $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$, $f(x) = \frac{1}{x\ln x}$

Sol $$\int_2^{\infty} \frac{1}{x\ln x}\,dx = \lim_{a\to\infty} \int_2^{a} \frac{dx}{x\ln x} = \lim_{a\to\infty} \left[\ln(\ln x)\right]_2^{a}$$

$$= \lim_{a\to\infty} \left[\ln(\ln a) - \ln(\ln 2)\right] = \infty \quad \text{diverge}$$

$\therefore \sum_{n=2}^{\infty} \frac{1}{n\ln n}$ diverges by integral test.

Test of convergence (Series with alternative signs).

If $\sum_{n=1}^{\infty} u_n$ is an infinite series with alternative signs and if (i) $|u_{n+1}| < |u_n|$, $\forall n$

(ii) $\lim_{n\to\infty} |u_n| = 0$

Then the series converges.

ex $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots$

sol- since $|u_{n+1}| < |u_n|$ $\left[\frac{1}{4} < \frac{1}{3}, \frac{1}{5} < \frac{1}{4}, \cdots\right]$ $n+1 > n$, $\frac{1}{n+1} < \frac{1}{n}$

and $\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore \sum \frac{(-1)^{n+1}}{n}$ converges

Absolute and Conditional Convergence

An infinite series with alternative signs $\sum u_n$ is said to be absolutely convergent if the corresponding series of absolute values $\sum |u_n|$ converges.

But if the series $\sum |u_n|$ diverges while the series $\sum u_n$ converges then $\sum u_n$ converges conditionally.

Note ① Every absolutely convergent series is convergent, but the converse is not true.

② Let $\sum u_n$ be a convergent series, then

ⓐ if $\sum |u_n|$ converges $\Rightarrow \sum u_n$ converges absolutely

ⓑ if $\sum |u_n|$ diverges $\Rightarrow \sum u_n$ converges conditionally

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ex $\sum \frac{(-1)^{n+1}}{n}$ converges by alternating series test

but $\sum |u_n| = \sum \frac{1}{n}$ diverges by P-test

$\therefore \sum u_n = \sum \frac{(-1)^{n+1}}{n}$ converges conditionally.

ex $\sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

sol $\sum |U_n| = \sum \frac{1}{n^2}$, $|U_n| = \frac{1}{n^2}$, $|U_{n+1}| = \frac{1}{(n+1)^2}$

(i) $(n+1)^2 > n^2 \Rightarrow \frac{1}{(n+1)^2} < \frac{1}{n^2} \quad \forall n \Rightarrow |U_{n+1}| < |U_n|$

(ii) $\lim_{n\to\infty} |U_n| = \lim_{n\to\infty} \frac{1}{n^2} = 0$

$\therefore \sum U_n = \sum \frac{(-1)^{n+1}}{n^2}$ converges

On the other hand

$\sum |U_n| = \sum \left|\frac{(-1)^{n+1}}{n^2}\right| = \sum \frac{1}{n^2}$ converges by P-test

$\therefore \sum U_n = \sum \frac{(-1)^{n+1}}{n^2}$ converges absolutely.

Ratio Test: The alternative series $\sum_{n=1}^{\infty} U_n$ converges absolutelly (and hence it is convergent) if $\lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right| < 1$ and diverges if $\lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right| > 1$ and if $\lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right| = 1$ the series may converges or diverges.

ex $\sum U_n = \sum \frac{2^n}{n!}$

sol $\lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}\right| = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$

$\therefore \sum \frac{2^n}{n!}$ converges absolutely and hence it converges.

ex $\frac{1}{5} + \frac{2}{6} + \frac{4}{7} + \frac{8}{8} + \frac{16}{9} + \cdots$ $\qquad U_n = \frac{2^{n-1}}{4+n}$

sol $\lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty} \left|\frac{2^n}{4+(n+1)} \cdot \frac{4+n}{2^{n-1}}\right| = \lim_{n\to\infty} \left|\frac{2(4+n)}{(5+n)}\right|$

$= 2 \lim_{n\to\infty} \frac{4+n}{5+n} = 2 > 1$

$\therefore$ The series diverges.

Limits that arise frequently

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0$$

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

$$\lim_{n\to\infty} \sqrt[n]{x} = 1 \quad , x > 0$$

$$\lim_{n\to\infty} x^n = 0 \quad , -1 < x < 1$$

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad , \text{all } x$$

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad , \text{all } x$$

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ex $\lim_{n\to\infty} 2^{1/n} = 2^{\lim_{n\to\infty} 1/n} = 2^0 = 1$

ex $\lim_{n\to\infty} x^{n+4} = \lim_{n\to\infty} x^n . x^4 = x^4 \lim_{n\to\infty} x^n = x^4 . 0 = 0 \quad , -1 < x < 1$

ex $\lim_{n\to\infty} \sqrt[n]{2n} = \lim_{n\to\infty} \sqrt[n]{2} \sqrt[n]{n} = 1 \cdot 1 = 1$

ex $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{2n} = \left[\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n\right]^2 = e^2$

ex $\lim_{n\to\infty} \sqrt[n]{3n+5}$

Let $a_n = \sqrt[n]{3n+5} = (3n+5)^{1/n} \Rightarrow \ln a_n = \ln (3n+5)^{1/n} = \frac{1}{n} \ln(3n+5)$

Now $\lim_{n\to\infty} (\ln a_n) = \lim_{n\to\infty} \frac{\ln(3n+5)}{n} = \lim_{n\to\infty} \frac{\frac{3}{3n+5}}{1} = 0$

$\Rightarrow a_n = e^0 = 1 \Rightarrow \lim_{n\to\infty} \sqrt[n]{3n+5} = 1$

or $\lim_{n\to\infty} (a_n) = e^{\lim_{n\to\infty} \ln a_n} = e^{\lim \frac{\ln(3n+5)}{n}} = e^0 = 1$

Examples: Test the following infinite Series.

① $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, $\sqrt{n} \leq n$, $\forall n$

$\frac{1}{\sqrt{n}} \geq \frac{1}{n} \Rightarrow \sum \frac{1}{\sqrt{n}} \geq \sum \frac{1}{n}$

$\because$ $\sum \frac{1}{n}$ diverges by P-Test

$\therefore$ $\sum \frac{1}{\sqrt{n}}$ diverges by Comparison test.

② $\sum_{1}^{\infty} \frac{1}{3n-2} = \frac{1}{3} \sum_{1}^{\infty} \frac{1}{n-\frac{2}{3}}$, $\because$ $n-\frac{2}{3} < n \Rightarrow \frac{1}{n-\frac{2}{3}} > \frac{1}{n}$

$\therefore \sum \frac{1}{n-\frac{2}{3}} > \sum \frac{1}{n}$

$\because$ $\sum \frac{1}{n}$ diverges by P-Test

$\therefore$ $\sum \frac{1}{n-2/3}$ diverges by comparison test

$\therefore$ $\frac{1}{3} \sum \frac{1}{n-2/3}$ " also.

③ $\sum_{1}^{\infty} \frac{2}{2^n+3}$

$2^n+3 > 2^n$, $\forall n \Rightarrow \frac{1}{2^n+3} < \frac{1}{2^n} \Rightarrow \sum \frac{2}{2^n+3} < \sum \frac{1}{2^{n-1}}$

$\because$ $\sum \frac{1}{2^{n-1}}$ converges since it is Geometric Series $|r| = \frac{1}{2} < 1$

$\therefore$ $\sum \frac{2}{2^n+3}$ converges by comparison test.

④ $\sum_{1}^{\infty} \frac{n}{n+2}$

$n+2 \leq 3n^2 \Rightarrow \frac{n}{n+2} \geq \frac{n}{3n^2} = \frac{1}{3n} \Rightarrow \sum \frac{n}{n+2} \geq \frac{1}{3} \sum \frac{1}{n}$

$\because$ $\sum \frac{1}{n}$ diverges by P-test

$\therefore$ $\sum \frac{n}{n+2}$ diverges by comp. test.

⑤ $\sum_{1}^{\infty} \frac{\sin^2 x}{2^n}$, $\sin x < 1 \Rightarrow \sin^2 x < 1 \Rightarrow \frac{\sin^2 x}{2^n} < \frac{1}{2^n} \Rightarrow \sum \frac{\sin^2 x}{2^n} < \sum \frac{1}{2^n}$

$\because$ $\sum \frac{1}{2^n}$ converges (geom. series, $|r| = \frac{1}{2} < 1$)

$\therefore$ $\sum \frac{\sin^2 x}{2^n}$ converges by comp. test.

⑥ $\sum_{1}^{\infty} \frac{1}{1+\ln n}$, $1+\ln n < n \Rightarrow \frac{1}{1+\ln n} > \frac{1}{n} \Rightarrow \sum \frac{1}{1+\ln n} > \sum \frac{1}{n}$

$\because \sum \frac{1}{n}$ diverges by P-test.

$\therefore \sum \frac{1}{1+\ln n}$ div. by comp. test.

⑦ $\sum \frac{1}{(2n+1)!}$, $(2n+1)! \geq n^2 \Rightarrow \frac{1}{(2n+1)!} \leq \frac{1}{n^2}$

$$\Rightarrow \sum \frac{1}{(2n+1)!} \leq \sum \frac{1}{n^2}$$

$\because \sum \frac{1}{n^2}$ is conv. by P-test.

$\therefore \sum \frac{1}{(2n+1)!}$ is conv. by comp. test.

⑧ $\sum \frac{1}{n\sqrt{n}}$ converges by P-test since $p = \frac{3}{2} > 1$

Or by integral test

$$\int_1^{\infty} \frac{dx}{x^{3/2}} = \lim_{a\to\infty} \int_1^a \frac{dx}{x^{3/2}} = \lim_{a\to\infty} \frac{x^{-1/2}}{-1/2}\Big]_1^a = -2\lim_{a\to\infty}\left(\frac{1}{\sqrt{a}}-1\right)$$

$$= -2\left[\lim_{a\to\infty}\frac{1}{\sqrt{a}} - 1\right] = 2 \quad \text{which is finite}$$

$\therefore \sum \frac{1}{n\sqrt{n}}$ converges by integral test.

⑨ $\sum_{1}^{\infty} n e^{-n^2}$

$$\int_1^{\infty} x e^{-x^2} dx = \lim_{a\to\infty}\int_1^a \frac{-2}{-2} x e^{-x^2} dx = -\frac{1}{2}\lim_{a\to\infty} e^{-x^2}\Big]_1^a$$

$$= -\frac{1}{2}\left[\lim_{a\to\infty} e^{-a^2} - e^{-1}\right] = -\frac{1}{2}\left[0 - e^{-1}\right] = \frac{1}{2e} \quad \text{finite}$$

$\therefore \sum n e^{-n^2}$ is conv.

⑩ $\sum_{1}^{\infty} \frac{\ln n}{2n^3-1}$, $\ln n < n$ and $2n^3-1 \geq n^3 \Rightarrow \frac{1}{2n^3-1} \leq \frac{1}{n^3}$

$$\Rightarrow \frac{\ln n}{2n^3-1} \leq \frac{n}{n^3} = \frac{1}{n^2} \Rightarrow \sum \frac{\ln n}{2n^3-1} \leq \sum \frac{1}{n^2}$$

$\because \sum \frac{1}{n^2}$ is conv. by P-test.

$\therefore \sum \frac{\ln n}{2n^3-1}$ is conv. by comp. test.

(11) $\sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)}$

$$\int_1^{\infty} \frac{dx}{(x+1)\ln(x+1)} = \lim_{a\to\infty} \{\ln\ln(x+1)\}_1^a = \lim_{a\to\infty} \ln(\ln(a+1)) - \ln\ln 2$$

$= \infty$ which is not finite

$\therefore \sum \frac{1}{(n+1)\ln(n+1)}$ diverges.

(12) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n}$, (i) $\because$ $n+1>n$ & $\ln(n+1)>\ln n$

$\therefore (n+1)\ln(n+1) > n\ln n$

$\therefore \frac{1}{(n+1)\ln(n+1)} < \frac{1}{n\ln n} \Rightarrow |u_{n+1}| < |u_n|$

(ii) $\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{n\ln n} = 0$

$\therefore \sum_2^{\infty} \frac{(-1)^n}{n\ln n}$ converges

But $\sum_2^{\infty} |u_n| = \sum \frac{1}{n\ln n}$ diverges by integral test (see ex. 11)

$\therefore \sum_2^{\infty} \frac{(-1)^n}{n\ln n}$ is conditionally convergent.

(13) $\sum_1^{\infty} \frac{(2n)!}{n^{100}}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \frac{(2n+2)!}{(n+1)^{100}} \cdot \frac{n^{100}}{(2n)!}$$

$$= \lim_{n\to\infty} (2n+2)(2n+1)\left(\frac{n}{n+1}\right)^{100} = \infty$$

$\therefore \sum \frac{(2n)!}{n^{100}}$ diverges by Ratio test.

(14) $\sum_1^{\infty} \frac{\cos n\pi}{n\sqrt{n}}$, $|\cos n\pi| \leq 1 \Rightarrow \frac{\cos n\pi}{n\sqrt{n}} \leq \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$

$\therefore \sum \left|\frac{\cos n\pi}{n\sqrt{n}}\right| \leq \sum \frac{1}{n^{3/2}}$ is convergent by comp. test

$\therefore \sum \frac{\cos n\pi}{n\sqrt{n}}$ is absolutely convergent.

(15) $\sum \frac{(-1)^n}{(2n+1)!}$, (i) $|u_{n+1}| < |u_n|$, (ii) $\lim_{n\to\infty} |u_n| = 0$

$\therefore \sum \frac{(-1)^n}{(2n+1)!}$ converges

$\because \sum |u_n| = \sum \frac{1}{(2n+1)!}$ (See ex. 7)

$\therefore \sum \frac{(-1)^n}{(2n+1)!}$ converges absoutely.

(16) $\sum \frac{2^n n!\, n!}{(2n)!}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \frac{2^{n+1} (n+1)!\,(n+1)!}{(2n+2)!} \; \frac{(2n)!}{2^n n!\, n!}$$

$$= 2 \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = 2 \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2} = \frac{1}{2} < 1$$

$\therefore \sum \frac{2^n n!\, n!}{(2n)!}$ converges by Ratio test.

(17) $\sum_1^\infty \frac{(n+3)!}{3!\, n!\, 3^n}$ converges because

$$\lim_{n\to\infty} \frac{(n+4)!}{3!\,(n+1)!\,3^{n+1}} \cdot \frac{3!\, n!\, 3^n}{(n+3)!} = \lim_{n\to\infty} \frac{n+4}{3(n+1)} = \frac{1}{3} < 1$$

(18) $\sum_1^\infty \ln \frac{n+2}{n+1}$

$= \sum_1^\infty [\ln(n+2) - \ln(n+1)]$

$= (\ln 3 - \ln 2) + (\ln 4 - \ln 3) + (\ln 5 - \ln 4) + \dots + (\ln(n+1) - \ln n) + (\ln(n+2) - \ln(n+1))$

$= -\ln 2 + \ln(n+2) \to \infty$ as $n \to \infty$

$\therefore \sum \ln \frac{n+2}{n+1}$ diverges

Power Series: Power Series are defined by

① $\sum_{n=k}^{\infty} C_n(x-a)^n$ or ② $\sum_{n=k}^{\infty} C_n x^n$ where $k \geqslant 0$

If they converge for some values of x, then there exist a function f for these values, such that,

①' $f(x) = \sum_{n=k}^{\infty} C_n(x-a)^n$ or ②' $f(x) = \sum_{n=k}^{\infty} C_n x^n$, $k \geqslant 0$

Theorem: Let $\sum_{n=k}^{\infty} C_n(x-a)^n$ any power series where $k \geqslant 0$, then either ① the series converges only where $x=a$,

or ② " " " for all x,

or ③ there exist a +ve constant R such that the series converges for all x where $|x-a|<R$ and diverges for all x where $|x-a|>R$, and may converges or diverges where $|x-a|=R$.

---

ex/ Find all values of x for which the given series converges. $\sum_{n=0}^{\infty} ((-1)^n n!\, x^n)/(10)^n$

sol/ By Ratio test $\lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right| < 1$ if the series is conv.

$\because$ The given series converges, then $\lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right| < 1$

$\therefore \lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)!\, x^{n+1}}{10^{n+1}} \cdot \frac{10^n}{(-1)^n n!\, x^n} \right|$

$= \lim_{n\to\infty} \left(\frac{n+1}{10}\right)|x| = \frac{|x|}{10} \lim_{n\to\infty} (n+1) = \infty$ which is greater that ONE

$\therefore \lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right|$ is Less than ONE only when $x=0$

$\therefore$ The Series Converges only when $x=0$ and diverges for all $x \neq 0$

Examples: Find all values of x for which the given series converge.

① $\sum_{1}^{\infty} \frac{nx^n}{2^n}$

sol: $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)\,x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n x^n}\right| < 1$

$\Rightarrow \lim_{n\to\infty} \left|\frac{n+1}{n} \frac{1}{2} x\right| = \left|\frac{x}{2}\right| \lim_{n\to\infty} \frac{n+1}{n} < 1$

or $\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2 \Rightarrow \boxed{-2 < x < 2}$

Now when $x = -2$ then $\sum u_n = \sum \frac{(-2)^n n}{2^n} = \sum (-1)^n n$ which is divergent, hence $\boxed{x \neq -2}$

when $x = 2 \Rightarrow \sum u_n = \sum \frac{2^n n}{2^n} = \sum n$ which is divergent also

$\therefore x \neq 2$

$\therefore x \in (-2, 2)$ or $-2 < x < 2$

② $\sum \frac{x^n}{n}$

sol: $\lim_{n\to\infty} \left|\frac{x^{n+1}}{n+1} \frac{n}{x^n}\right| = \lim_{n\to\infty} \left| x \cdot \frac{n}{n+1}\right| = |x| \lim_{n\to\infty} \frac{n}{n+1} = |x| < 1$

$\therefore -1 < x < 1$

when $x = -1 \Rightarrow \sum \frac{(-1)^n}{n}$ which is cond. conv. $\Rightarrow x = -1$

when $x = 1 \Rightarrow \sum \frac{1}{n}$ which is div. $\Rightarrow x \neq 1$

$\therefore x \in [-1, 1)$ or $-1 \leq x < 1$

③ $\sum_{1}^{\infty} \frac{(-1)^n (x+1)^n}{2^n n^2}$

sol $\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x+1)^{n+1}}{2^{n+1}(n+1)^2} \cdot \frac{2^n n^2}{(-1)^n (x+1)^n} \right| = \frac{1}{2}|x+1| \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2$

$= \frac{1}{2}|x+1| < 1 \Rightarrow -2 < x+1 < 2 \Rightarrow -3 < x < 1$

Now, when $x=-3 \Rightarrow \sum U_n = \sum \frac{(-1)^n(-2)^n}{2^n n^2} = \sum \frac{1}{n^2}$ which is convergent by p-test $\Rightarrow x=-3$

when $x=1 \Rightarrow \sum U_n = \sum \frac{(-1)^n 2^n}{2^n n^2} = \sum \frac{(-1)^n}{n}$ which is absolutely convergent $\Rightarrow x=1 \Rightarrow -3 \leq x \leq 1$

④ $\sum_{1}^{\infty} \frac{x^{2n+1}}{n^2}$

sol $\lim_{n\to\infty} \left| \frac{x^{2n+3}}{(n+1)^2} \cdot \frac{n^2}{x^{2n+1}} \right| = \lim_{n\to\infty} \left| x^2 \left(\frac{n}{n+1}\right)^2 \right| = |x^2| \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 < 1$

$\Rightarrow |x| < 1 \Rightarrow -1 < x < 1$

when $x=-1 \Rightarrow \sum U_n = \sum \frac{(-1)^{2n+1}}{n^2} = \sum \frac{-1}{n^2}$ converges by P-test

when $x=1 \Rightarrow \sum U_n = \sum \frac{1}{n^2}$ converges by P-test also

$\therefore -1 \leq x \leq 1$

---

⑤ $\sum_{0}^{\infty} \frac{(-1)^n x^n}{n!}$

sol

$\lim_{n\to\infty} \left| \frac{U_{n+1}}{U_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{(-1)^n x^n} \right| = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$

$\therefore$ The series converges for all x

Taylor's and Maclaurin's Series

Suppose there is a convergent series in an interval, then we defined a function in the same interval such that $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ for $|x-a| < R$

where $a_n = \frac{f^{(n)}(a)}{n!}$ where $f^{(n)}(a)$ is the n-th derivative of $f(x)$ at $x=a$

Hence

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

or $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, the R.H.S. is called the Taylor Series for the function f at $x=a$ (or the expansion of $f(x)$ at $x=a$).

The Maclaurin Series is a special case of Taylor series when $a=0$

---

ex Find the Maclaurin series generated by the expontial function $f(x) = e^x$

sol

$$f(x) = e^x \implies f(0) = e^0 = 1$$

$$f'(x) = e^x \implies f'(0) = e^0 = 1$$

$$f''(x) = e^x \implies f''(0) = e^0 = 1$$

$$\vdots$$

$$f^{(n)}(x) = e^x \implies f^{(n)}(0) = e^0 = 1$$

$$\therefore f(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n + \dots$$

$$\therefore e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Similarly $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$ (H.W.)

ex Find the Maclaurin series for $\sin x$.

sol. $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$f(x) = \sin x \Rightarrow f(0) = 0$

$f'(x) = \cos x \Rightarrow f'(0) = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$

$\vdots \qquad \vdots$

$\therefore f(x) = 0 + \frac{x}{1!} + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$

$\therefore \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Similarly $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ (H.W.)

---

ex Find the Maclaurin expansion for the Binomial series $f(x) = (1+x)^\alpha$ where α is any real no.

sol $f(x) = (1+x)^\alpha \Rightarrow f(0) = 1$

$f'(x) = \alpha(1+x)^{\alpha-1} \Rightarrow f'(0) = \alpha$

$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2} \Rightarrow f''(0) = \alpha(\alpha-1)$

$f'''(x) = \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3} \Rightarrow f'''(0) = \alpha(\alpha-1)(\alpha-2)$

$\vdots$

$f^{(n)}(0) = \alpha(\alpha-1)(\alpha-2)\dots(\alpha-(n+1))$

$\therefore f(x) = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \dots + \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} x^n + \dots$

or $(1+x)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha!}{n!(\alpha-n)!} x^n$

Note that $\alpha! = \alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)(\alpha-n)(\alpha-n-1)\dots 3\cdot 2\cdot 1$

ex Find the Taylor's expansion for $\frac{1}{x}$ when $x=1$

sol. $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$

Now,

$f(x) = x^{-1} \implies f(1) = 1$

$f'(x) = (-1)x^{-2} \implies f'(1) = -1$

$f''(x) = (-1)(-2)x^{-3} \implies f''(1) = (-1)^2\, 2!$

$f'''(x) = (-1)(-2)(-3)x^{-4} \implies f'''(1) = (-1)^3\, 3!$

$\vdots$

$f^{(n)}(x) = (-1)(-2)\cdots(-n)x^{-n-1} \implies f^{(n)}(1) = (-1)^n\, n!$

$\therefore f(x) = 1 - \frac{(x-1)}{1!} + \frac{2!(x-1)^2}{2!} - \frac{3!(x-1)^3}{3!} + \frac{4!\,(x-1)^4}{4!} + \cdots + \frac{(-1)^n n!\,(x-1)^n}{n!} + \cdots$

$\therefore \frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$

---

ex Find the Taylor's expansion for $\ln x$ when $x=3$

sol. $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n$

Now,

$f(x) = \ln x \implies f(3) = \ln 3$

$f'(x) = x^{-1} \implies f'(3) = 3^{-1} = (-1)^0\, 0!\; 3^{-1}$

$f''(x) = (-1)x^{-2} \implies f''(3) = (-1)\, 3^{-2} = (-1)\, 1!\; 3^{-2}$

$f'''(x) = (-1)(-2)x^{-3} \implies f'''(3) = (-1)^2\, 2!\; x^{-3}$

$f''''(x) = (-1)(-2)(-3)x^{-4} \implies f''''(3) = (-1)^3\, 3!\; x^{-4}$

$\vdots$

$f^{(n)}(x) = (-1)(-2)(-3)\cdots(-(n-1))x^{-n} \implies f^{(n)}(3) = (-1)^{n-1}(n-1)!\; x^{-n}$

$\therefore f(x) = \ln 3 + \frac{1}{3.1!}(x-3) - \frac{1!}{3^2.2!}(x-3)^2 + \frac{2!}{3^3.3!}(x-3)^3 + \cdots + \frac{(-1)^{n-1}(n-1)!}{3^n\, n!}(x-3)^n + \cdots$

$\therefore \ln x = \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n.\, n}(x-3)^n$

Prove the following Maclaurin Series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n \quad , |x|<1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^n x^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n \quad , |x|<1$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \quad , -1<x\leq 1$$

$$\ln\left(\frac{1+x}{1-x}\right) = 2\tanh^{-1} x = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots\right)$$

$$= 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \quad , |x|<1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^{2n-1}}{2n-1} \quad , |x|\leq 1$$

$$\sinh x = x + \frac{x^3}{3!} + \cdots + \frac{x^{2n-1}}{(2n-1)!} + \cdots = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

Differentiation and Integration for infinite Series

If $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$, for $|x-a|<R$

then $f'(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$

and $\int f(x)\,dx = \sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1}$

① Find the expansion, in power of x, for e^{-x^2}, $\forall x$.

$$\because \quad e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad , \quad \forall x$$

Put $x \longrightarrow -x^2$

$$\therefore \quad e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

$$= 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots ,$$

② Find the expansion, in powers of x, for $\frac{1}{1+x^2}$ and $\frac{x}{(1+x^2)^2}$

$$\because \quad \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \quad , \quad \text{for } |x| < 1$$

Put $x \longrightarrow x^2$

$$\therefore \quad \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n (x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad \text{—— ①}$$ الطلب الأول

لإيجاد الطلب الثاني نشتق طرفي
المعادلة ① بالنسبة لـ x

$$-(1+x^2)^{-2} \cdot 2x = \sum_{n=1}^{\infty} (-1)^n \, 2n \, x^{2n-1}$$

$$-\frac{\cancel{2}x}{(1+x^2)^2} = \cancel{2} \sum_{n=1}^{\infty} (-1)^n \, n \, x^{2n-1} \qquad \div(-2)$$

$$\frac{x}{(1+x^2)^2} = -\sum_{n=1}^{\infty} (-1)^n \, n \, x^{2n-1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \, n \, x^{2n-1}$$ الطلب الثاني

(3) Find the expansion, in Power of x, for $\tan^{-1}x$ $\forall x$ such that $|x|<1$, and find the value of π.

Since $\displaystyle\int_0^x \frac{dx}{1+x^2} = \tan^{-1}x$

$$\because \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n , \quad \text{for } |x|<1$$

Put $x \to x^2$

$$\therefore \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

نأخذ تكامل الطرفين بالنسبة الى x من 0 الى x نحصل

$$\int_0^x \frac{dx}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n \int_0^x x^{2n}\,dx$$

$$\left[\tan^{-1}x\right]_0^x = \sum_{n=0}^{\infty} (-1)^n \left[\frac{x^{2n+1}}{2n+1}\right]_0^x$$

$$\tan^{-1}x - \underbrace{\tan^{-1}0}_{=0} = \sum_{n=0}^{\infty} (-1)^n \left[\frac{x^{2n+1}}{2n+1} - 0\right]$$

$$\therefore \tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \quad (1)$$

الطلب الاول

الآن لايجاد قيمة π نأخذ الغاية
لطرفي المعادلة (1) عندما $x \to 1$ نحصل

$$\lim_{x\to 1} \tan^{-1}x = \lim_{x\to 1}\left[x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right]$$

$$\tan^{-1}1 = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$

$$\therefore \pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right)$$

$$\pi \simeq 3.14$$

(4) Find the expansion, in Power of x, for $\ln(1+x)$?

$$\because \left[\ln(1+x)\right] = \int_0^x \frac{1}{1+x}\,dx$$

لذلك نوجد متسلسلة $\frac{1}{1+x}$ ثم نكامل الطرفين بالنسبة الى x .

$$\because \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \quad , \quad |x| < 1$$

$$\int_0^x \frac{dx}{1+x} = \sum_{n=0}^{\infty} (-1)^n \int_0^x x^n\,dx$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

---

(6) Find the expansion, in Power of x, for $\frac{e^x - 1}{x}$ and by using the result show that $\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$.

$$\because e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^x - 1 = \cancel{1} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots - \cancel{1} \qquad \div x$$

$$\therefore \frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$$

للحصول على ~~المطلب الثاني~~ نشتق الطرفين بالنسبة الى x فنحصل

$$\frac{x(e^x) - (e^x - 1)\cdot 1}{x^2} = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{(n+1)!}$$

بوضع $x = 1$ في الطرفين نحصل على

$$\frac{\cancel{e} - \cancel{e} + 1}{1^2} = \sum_{n=1}^{\infty} \frac{n\,(1)^{n-1}}{(n+1)!}$$

$$\therefore \sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$$

سؤال الامتحان

Obtain the expansion

$$\tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots, \quad \forall x > 1$$

$$\because \quad \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \quad , \forall \quad |x|<1$$

Put $x \longrightarrow \frac{1}{x^2}$

$$\frac{1}{1+\frac{1}{x^2}} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{x^2}\right)^n \quad , \quad \frac{1}{|x^2|} < 1 \Rightarrow$$

$$\frac{x^2}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{-2n} \quad , \quad \frac{1}{x} < 1 \Longrightarrow x > 1$$

$$\left(1 - \frac{1}{1+x^2}\right) = \sum_{n=0}^{\infty} (-1)^n x^{-2n} \quad , \quad x > 1$$

$$\begin{array}{r|l} & 1 \\ x^2+1 & \cancel{x^2} \\ & \mp \cancel{x^2} \mp 1 \\ \hline & -1 \end{array}$$

بأخذ التكامل للطرفين بالنسبة لـ x نحصل

$$\int \left(1 - \frac{1}{1+x^2}\right) dx = \sum_{n=0}^{\infty} (-1)^n \int x^{-2n}\, dx$$

$$x - \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{-2n+1}}{-2n+1} + \overset{\text{ثابت التكامل}}{C} \quad , \quad x > 1$$

$$\cancel{x} - \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{(2n-1)\, x^{2n-1}} + C$$

$$= \overset{0}{\cancel{x}} + \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \cdots + C$$

لإيجاد الثابت C نأخذ الغاية للطرفين عندما $x \to \infty^+$

$$\lim_{x\to\infty^+} \left[-\tan^{-1} x\right] = \lim_{x\to\infty^+} \left[\frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \cdots + C\right]$$

$$-\frac{\pi}{2} = 0 + C \Longrightarrow C = -\frac{\pi}{2}$$

$$\therefore -\tan^{-1} x = -\frac{\pi}{2} + \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \cdots \qquad \times -1$$

$$\therefore \tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots, \quad \forall x > 1$$

(7) Find the expansion, in power of x, for $x^2 e^{-x}$ and by using the result to show that $\sum_{n=1}^{\infty} (-2)^{n+1} \frac{n+2}{n!} = 4$.

$$\because \quad e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Put $x \longrightarrow -x$

$$\therefore \quad e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

نضرب الطرفين في x^2 نحصل ع

$$x^2 e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+2}}{n!}$$

نشتق الطرفين بالنسبة الى x نحصل ع

$$2x e^{-x} - x^2 e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{(n+2)\, x^{n+1}}{n!}$$

نوضع $x=2$ في الطرفين نحصل ع

$$\cancel{4e^{-2}} - \cancel{4e^{-2}} = \sum_{n=0}^{\infty} (-1)^n \frac{(n+2)\, 2^{n+1}}{n!}$$

$$0 = \sum_{n=0}^{\infty} (-1)^n \frac{(n+2)\, 2^{n+1}}{n!} \quad \times -1$$

$$0 = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(n+2)\, 2^{n+1}}{n!}$$

$$0 = \sum_{n=0}^{\infty} (-2)^{n+1} \frac{(n+2)}{n!}$$

$$0 = -4 + \sum_{n=1}^{\infty} (-2)^{n+1} \frac{n+2}{n!}$$

$$4 = \sum_{n=1}^{\infty} (-2)^{n+1} \frac{n+2}{n!}$$

⑧ Find the expansion, in powers of x, for $\ln\left(\frac{1+x}{1-x}\right)$ such that $|x|<1$

since $\quad \ln(1+x) = \int_0^x \frac{dx}{1+x}$

$$\therefore \quad \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \quad , \quad |x|<1$$

بأخذ التكامل للطرفين

$$\int_0^x \frac{dx}{1+x} = \sum_{n=0}^{\infty} (-1)^n \int_0^x x^n \, dx$$

$$\ln(1+x) = \sum_0^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \quad \text{———} \quad (1)$$

Now,

Put $x \longrightarrow -x$

$$\ln(1-x) = \sum_{n=0}^{\infty} (-1)^n \frac{(-x)^{n+1}}{n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n (-1)^{n+1} \frac{x^{n+1}}{n+1}$$

$(-1)^{2n}(-1) = 1 \times -1$

$$= -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \quad \text{———} \quad (2)$$

Now,

$$\therefore \quad \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

$$= \sum_0^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + \sum_0^{\infty} \frac{x^{n+1}}{n+1}$$

$$= \left(x - \cancel{\frac{x^2}{2}} + \frac{x^3}{3} - \cancel{\frac{x^4}{4}} + \cdots\right) + \left(x + \cancel{\frac{x^2}{2}} + \frac{x^3}{3} + \cancel{\frac{x^4}{4}} \cdots\right)$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$

$$= 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$