Differential Equations

A ***Differential Equation*** is an equation that contains one or more derivatives of a differentiable function. An equation with partial derivatives is called a ***Partial Differential Equation***. While, an equation with ordinary derivatives, that is, derivatives of a function of a single variable, is called an ***Ordinary Differential Equation***.

The ***order*** of a differential equation is the order of the equation's highest order derivative. A differential equation is ***linear*** if it can be put in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x)$$

The ***degree*** of a differential equation is the power (exponent) of the equation's highest order derivative.

Example

First order, first degree, linear $\quad \frac{dy}{dx} = 5y, \qquad 3\frac{dy}{dx} - \sin x = 0$

Third order, second degree, nonlinear $\quad \left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^5 - \frac{dy}{dx} = e^x$

Solution of First Order Differential Equations

1) Separable Equations

A first order differential equations is separable if it can be put in the form

$$M(x)dx + N(y)dy = 0$$

Steps for Solving a Separable First Order Differential Equation

i. Write the equation in the form $M(x)dx + N(y)dy = 0$.

ii. Integrate M with respect to x and N with respect to y to obtain an equation that relates y and x.

Example

Solve the following differential equations

(a) $\dfrac{dy}{dx} = (1+y^2)e^x$, (b) $\dfrac{dy}{dx} = \dfrac{x(2\ln x + 1)}{\sin y + y\cos y}$, (c) $e^{x+y}dx = \dfrac{dy}{x}$

Solution

(a) $\dfrac{dy}{dx} = (1+y^2)e^x \quad \Rightarrow \quad e^x dx - \dfrac{1}{1+y^2}dy = 0$

$$\int e^x dx - \int \frac{1}{1+y^2}dy = C \quad \Rightarrow \quad e^x - \tan^{-1} y = C$$

$$\tan^{-1} y = e^x - C \quad \Rightarrow \quad y = \tan(e^x - C)$$

(b) $\dfrac{dy}{dx} = \dfrac{x(2\ln x + 1)}{\sin y + y\cos y} \quad \Rightarrow \quad (\sin y + y\cos y)dy = x(2\ln x + 1)dx$

$$\int(\sin y + y\cos y)dy - \int x(2\ln x + 1)dx = C$$

$$\int \sin(y)dy + \int y\cos(y)dy - 2\int x\ln(x)dx - \int x dx = C$$

$$-\cos(y) + \left[y\sin y - \int \sin(y)dy\right] - 2\left[\ln(x)\times\frac{x^2}{2} - \int \frac{x^2}{2}\times\frac{1}{x}dx\right] - \frac{x^2}{2} = C$$

$$-\cos(y) + y\sin y + \cos y - x^2\ln x + \frac{x^2}{2} - \frac{x^2}{2} = C$$

$$y\sin y - x^2\ln x = C.$$

(c) $e^{x+y}dx = \dfrac{dy}{x} \quad \Rightarrow \quad e^{x}e^{y}dx = \dfrac{dy}{x}$

$$xe^{x}dx = \frac{dy}{e^{y}} \Rightarrow xe^{x}dx - e^{-y}dy = 0 \Rightarrow \quad \int xe^{x}dx - \int e^{-y}dy = C$$

$$xe^{x} - \int e^{x}dx + e^{-y} = C \quad \Rightarrow \quad xe^{x} - e^{x} + e^{-y} = C$$

<u>*Notes*</u>

1) $f_1(x)g_1(y)dy + f_2(x)g_2(y)dx = 0$ *Separable*

2) $\dfrac{f_1(x)}{g_1(y)}dy + \dfrac{f_2(x)}{g_2(y)}dx = 0$ *Separable*

3) $[f_1(x) \pm g_1(y)]dy + [f_2(x) \pm g_2(y)]dx$ *Not Separable*

<u>Example</u>

$$f_1(x) = x, \quad f_2(x) = \sin(x), \quad g_1(y) = y, \quad g_2(y) = \tan(y)$$

1) $xydy + \sin(x)\tan(y)dx = 0 \quad \Rightarrow \quad xydy = -\sin(x)\tan(y)dx$

$$\frac{y}{\tan(y)}dy = -\frac{\sin(x)}{x}dx$$ *Separable*

2) $\dfrac{x}{y}dy + \dfrac{\sin(x)}{\tan(y)}dx = 0 \quad \Rightarrow \quad \dfrac{x}{y}dy = -\dfrac{\sin(x)}{\tan(y)}dx$

$$\frac{\tan(y)}{y}dy = -\frac{\sin(x)}{x}dx$$ *Separable*

3) $(x+y)dy + (\sin(x) + \tan(y))dx = 0$

$$(x+y)dy = -(\sin(x) + \tan(y))dx$$ *Not Separable*

Special Type of Separable Equations

If $\frac{dy}{dx} = f(ax+by+c)$; then let $z = ax+by+c$ and the resultant equation may be reduced to a separable equation.

Example

Solve the differential equation $\frac{dy}{dx} = \tan^2(x+y)$.

Solution

$$z = x+y \rightarrow dz = dx + dy \quad \Rightarrow \quad \frac{dz}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{dz}{dx} - 1 \quad \Rightarrow \quad \frac{dz}{dx} - 1 = \tan^2(z)$$

$$\frac{dz}{dx} = \tan^2(z) + 1 \quad \Rightarrow \quad \frac{dz}{dx} = \sec^2(z)$$

$$\frac{dz}{\sec^2(z)} = dx \quad \Rightarrow \quad \cos^2(z)dz = dx$$

$$\int \cos^2(z)dz - \int dx = C \quad \Rightarrow \quad \int \frac{1+\cos(2z)}{2} dz - \int dx = C$$

$$\frac{1}{2}z + \frac{1}{4}\sin(2z) - x = C$$

While $z = x+y$, then the solution is

$$\frac{1}{2}(x+y) + \frac{1}{4}\sin(2(x+y)) - x = C$$

Note

For the differential equation $\frac{dy}{dx} = \tan(x+y) - \sec(x-y)$, we can not use the assumption because the difference between the arguments of $\tan(x+y)$ and $\sec(x-y)$. So, the differential equation can not be converted to a separable equation.

Exercises

Find the solution of the following Differential Equations

1) $y' = ky$

2) $y' = -xy$

3) $y' - 2y + a = 0$

4) $xy' + by = 0$

5) $(x \ln x)y' = y$

6) $(x+2)y' - xy = 0$

7) $y' = 2x^{-1}\sqrt{y-1}$

8) $yy' = 2x\exp(y^2)$

9) $2y' = y\cot(x)$

10) $y' + \csc(y) = 0$

11) $y' = (1+x)(1+y^2)$

12) $yy' = 0.5\sin^2(\omega x)$

13) $y' = y\tanh(x)$

14) $y'\sin(2x) = y\cos(2x)$

15) $y' = y\tan(2x),\ y(0) = 2$

16) $(x+1)y' = 2y,\ y(0) = 1$

17) $2xy' = 3y,\ y(1) = 4$

18) $y'x\ln(x) = y,\ y(2) = \ln(4)$

19) $y' = 2e^x y^3,\ y(0) = 0.5$

20) $(x^2+1)yy' = 1,\ y(0) = -3$

21) $dr\sin(\theta) = 2r\cos(\theta)d\theta, r(\pi/2) = 2$

22) $\upsilon\, d\upsilon/dx = C,\ \upsilon(x_0) = \upsilon_0$

Homogeneous Function

If $f(\lambda x, \lambda y) = \lambda^n f(x, y)$ then, $f(x, y)$ is homogeneous function and n represents the degree of the homogeneous function.

Example

For the function $f(x, y) = x^2 + y^2$ then

$$f(\lambda x, \lambda y) = (\lambda x)^2 + (\lambda y)^2$$
$$= \lambda^2 x^2 + \lambda^2 y^2$$
$$= \lambda^2 (x^2 + y^2) = \lambda^2 f(x, y)$$

So, the function $f(x, y)$ is homogeneous with degree 2.

Example

For the function $f(x, y) = x + y^2$ then

$$f(\lambda x, \lambda y) = \lambda x + (\lambda y)^2$$
$$= \lambda x + \lambda^2 y^2$$
$$= \lambda (x + \lambda y^2)$$

So, the function $f(x, y)$ is not homogeneous.

Example

$f(x, y) = x^2 + y^2 + 5$ (Non-homogeneous)

$f(x, y) = x^3 + xy + x$ (Non-homogeneous)

$f(x, y) = \cos(xy)$ (Non-homogeneous)

$f(x, y) = \cos(x^2 \pm y^2)$ (Non-homogeneous)

$$f(x,y)=\cos\left(\frac{x}{y}\right) \qquad \text{(Homogeneous)}$$

$$f(x,y)=\cos\left(\frac{x^2}{y}\right) \qquad \text{(Non-homogeneous)}$$

Homogeneous Equations

The differential equation $M(x,y)dx+N(x,y)dy$ is homogeneous if M and N are homogeneous functions of the same degree.

Example

1) $(x^2+y^2)dx+xydy=0$

This is homogeneous because M and N are both homogeneous with degree 2.

2) $(x^3+y^3)dx+xydy=0$

This is not homogeneous because M is homogeneous with degree 3 while N is homogeneous with degree 2.

3) $xdx+(x^2+y)dy=0$

This is not homogeneous because N is not homogeneous.

Solution of Homogeneous Equations

A homogeneous first order differential equation can be put in the form

$$\frac{dy}{dx}=F\left(\frac{y}{x}\right)$$

This equation can be changed into separable equation with the substitutions

$$v = \frac{y}{x} \quad \Rightarrow \quad y = vx \quad \Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Then becomes $$v + x\frac{dv}{dx} = F(v)$$

which can be rearranged algebraically to give

$$\frac{dx}{x} + \frac{dv}{v - F(v)} = 0$$

with the variables now separated, the equation can now be solved by integrating with respect to x and v. We can then return to x and y by substituting $v = y/x$.

Example

Find the solution of the differential equation

$$\frac{dy}{dx} = -\frac{x^2 + y^2}{2xy}$$

that satisfies the condition $y(1) = 1$.

Solution

Dividing the numerator and denominator of the right-hand side by x^2 gives

$$\frac{dy}{dx} = -\frac{1 + (y/x)^2}{2(y/x)} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{1 + v^2}{2v} = F(v)$$

$$v - F(v) = v + \frac{1 + v^2}{2v} = \frac{2v^2 + 1 + v^2}{2v} = \frac{3v^2 + 1}{2v}$$

$$\frac{dx}{x} + \frac{dv}{v - F(v)} = 0 \quad \Rightarrow \quad \frac{dx}{x} + \frac{2v\,dv}{3v^2 + 1} = 0$$

The solution of this equation can be written as

$$\int \frac{dx}{x} + \int \frac{2v\,dv}{3v^2+1} = C \quad \Rightarrow \quad \ln x + \frac{1}{3}\ln(1+3v^2) = C$$

$$3\ln x + \ln(1+3v^2) = 3C \quad \Rightarrow \quad \ln x^3 + \ln(1+3v^2) = 3C$$

$$e^{\ln x^3 + \ln(1+3v^2)} = e^{3C} \quad \Rightarrow \quad e^{\ln x^3} \times e^{\ln(1+3v^2)} = e^{3C}$$

$$x^3(1+3v^2) = C' \quad \Rightarrow \quad x^3\left(1+3\frac{y^2}{x^2}\right) = C'$$

$$x^3 + 3xy^2 = C'$$

The condition is that when $x = 1$ then $y = 1$ and the constant C' can be found

$$(1)^3 + 3(1)(1)^2 = C' \quad \Rightarrow \quad C' = 4$$

The final solution is $x^3 + 3xy^2 = 4$.

Reducible to Homogeneous

If the differential equation has the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

Case 1: if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$ then $z = a_1x + b_1y$

Case 2: if $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$ then intersect the two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ to find the intersection point (h, k) and let

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$$x = X + h \quad \Rightarrow \quad dx = dX, \text{ and } \quad y = Y + k \quad \Rightarrow \quad dy = dY$$

Example

Solve the differential equation $\dfrac{dx}{dy}=\dfrac{4x+6y+5}{3y+2x+4}$

Solution

$$\frac{dy}{dx}=\frac{2x+3y+4}{4x+6y+5}$$

$$a_1=2,\quad a_2=4 \quad\Rightarrow\quad \frac{a_1}{a_2}=\frac{2}{4}=\frac{1}{2}$$

$$b_1=3,\quad b_2=6 \quad\Rightarrow\quad \frac{b_1}{b_2}=\frac{3}{6}=\frac{1}{2}$$

So $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2} \quad\Rightarrow\quad$ ***Case 1***

Let $z=2x+3y \quad\Rightarrow\quad \dfrac{dz}{dx}=2+3\dfrac{dy}{dx}$

$$\frac{dy}{dx}=\frac{1}{3}\left(\frac{dz}{dx}-2\right) \quad\Rightarrow\quad \frac{1}{3}\frac{dz}{dx}-\frac{2}{3}=\frac{z+4}{2z+5} \quad\Rightarrow\quad \frac{dz}{dx}=\frac{3z+12}{2z+5}+2$$

$$\frac{dz}{dx}=\frac{3z+12+4z+10}{2z+5} \quad\Rightarrow\quad \frac{dz}{dx}=\frac{7z+22}{2z+5} \quad\Rightarrow\quad \frac{2z+5}{7z+22}dz=dx$$

$$\int\frac{2z+5}{7z+22}dz-\int dx=C \quad\Rightarrow\quad \int\left(\frac{2}{7}-\frac{9}{7}\times\frac{1}{7z+22}\right)dz-\int dx=\cancel{C}$$

$$\int\frac{2}{7}dz-\int\frac{9}{7\times 7}\times\frac{7}{7z+22}dz-\int dx=C$$

$$\frac{2}{7}z-\frac{9}{49}\times\ln(7z+22)-x=C$$

$$\frac{2}{7}(2x+3y)-\frac{9}{49}\times\ln\big(7(2x+3y)+22\big)-x=$$

Example

Solve the differential equation $(2x+y-3)dy=(x+2y-3)dx$

Solution

$$\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$$

$$a_1=1,\quad a_2=2 \qquad \Rightarrow \qquad \frac{a_1}{a_2}=\frac{1}{2}$$

$$b_1=2,\quad b_2=1 \qquad \Rightarrow \qquad \frac{b_1}{b_2}=2$$

So $\quad \dfrac{a_1}{a_2}\neq\dfrac{b_1}{b_2} \qquad \Rightarrow \qquad$ ***Case 2***

$$\begin{array}{llllll}
x & +2y & -3 & =0 & \ldots\ldots & (1)\\
2x & +y & -3 & =0 & \ldots\ldots & (2)\\
\hline
+2x & +4y & -6 & =0 & & \\
\mp 2x & \mp y & \pm 3 & =0 & & \\
\hline
 & 3y & -3 & =0 & &
\end{array}$$

$$\Rightarrow \quad y=1$$

Substituting into (2), we get

$$2x+1-3=0 \quad \Rightarrow \quad x=1$$

The intersection point $(h,k)=(1,1)$.

Let $\quad x=X+1 \quad \Rightarrow \quad dx=dX$

$\quad y=Y+1 \quad \Rightarrow \quad dy=dY$

$$\frac{dY}{dX}=\frac{(X+1)+2(Y+1)-3}{2(X+1)+(Y+1)-3}$$

$$=\frac{X+1+2Y+2-3}{2X+2+Y+1-3} \quad \Rightarrow \quad \frac{dY}{dX}=\frac{X+2Y}{2X+Y}$$

$$\frac{dY}{dX} = \frac{1+2\frac{Y}{X}}{2+\frac{Y}{X}}$$

Let $\quad v = \frac{Y}{X} \quad \Rightarrow \quad \frac{dY}{dX} = \frac{1+2v}{2+v} = F(v)$

$$\frac{dX}{X} + \frac{dv}{v - F(v)} = 0$$

$$v - F(v) = v - \frac{1+2v}{2+v}$$

$$= \frac{2v + v^2 - 1 - 2v}{2+v} = \frac{v^2-1}{2+v}$$

$$\int \frac{dX}{X} + \frac{2+v}{v^2-1}dv = 0 \quad \Rightarrow \quad \ln X + \int \frac{2+v}{v^2-1}dv = C$$

$$\frac{2+v}{v^2-1} = \frac{A}{v+1} + \frac{B}{v-1} = \frac{A(v-1)+B(v+1)}{v^2-1}$$

$$2+v = A(v-1)+B(v+1) \Rightarrow 2+v = Av - A + Bv + B$$

At $\quad v = 1 \quad \Rightarrow \quad B = \frac{3}{2}$

At $\quad v = -1 \quad \Rightarrow \quad A = \frac{-1}{2}$

$$\ln X + \int \frac{-1/2}{v+1}dv + \int \frac{3/2}{v-1}dv = C$$

$$\ln X - \frac{1}{2}\ln(v+1) + \frac{3}{2}\ln(v-1) = C$$

$$\ln X - \frac{1}{2}\ln\left(\frac{Y}{X}+1\right) + \frac{3}{2}\ln\left(\frac{Y}{X}-1\right) = C$$

$$\ln(x-1) - \frac{1}{2}\ln\left(\frac{(y-1)}{(x-1)}+1\right) + \frac{3}{2}\ln\left(\frac{(y-1)}{(x-1)}-1\right) = C$$

Exercises

Find the solution of the following Differential Equations

1) $xy' = x + y$

2) $xy' = 2x + 2y$

3) $x^2y' = y^2 + xy + x^2$

4) $x^2y' = y^2 + 5xy + 4x^2$

5) $y' = \dfrac{y-x}{y+x}$

6) $y' = \dfrac{y+x}{y-x}$

7) $y' = (y-x)^2$

8) $y' = \tan(x+y) - 1$

9) $y' = \dfrac{y-x+1}{y-x+5}$

10) $y' = \dfrac{1-2y-4x}{1+y+2x}$

2) Linear First Order Equations

A differential equation that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are functions of x, is called a ***Linear First Order Equation***. The solution is

$$y = \frac{1}{\rho(x)}\int \rho(x)Q(x)dx$$

where $$\rho(x) = e^{\int P(x)dx}$$

Steps for Solving a Linear First Order Equation

i. Put it in standard form and identify the functions P and Q.

ii. Find an anti-derivative of $P(x)$.

iii. Find the integrating factor $\rho(x) = e^{\int P(x)dx}$.

iv. Find y using the following equation

$$y = \frac{1}{\rho(x)} \int \rho(x) Q(x) dx$$

Example

Solve the equation $x\dfrac{dy}{dx} - 3y = x^2$

Solution

Step 1: ***Put the equation in standard form and identify the functions*** P ***and*** Q. To do so, we divide both sides of the equation by the coefficient of dy/dx, in this case x, obtaining

$$\frac{dy}{dx} - \frac{3}{x}y = x \qquad \Rightarrow \qquad P(x) = -\frac{3}{x}, \qquad Q(x) = x.$$

Step 2: ***Find an anti-derivative of*** $P(x)$.

$$\int P(x)dx = \int -\frac{3}{x}dx = -3\int \frac{1}{x}dx = -3\ln(x)$$

Step 3: ***Find the integrating factor*** $\rho(x)$.

$$\rho(x) = e^{\int P(x)dx} = e^{-3\ln x} = e^{\ln x^{-3}} = e^{\ln \frac{1}{x^3}} = \frac{1}{x^3}$$

Step 4: *Find the solution.*

$$y = \frac{1}{\rho(x)} \int \rho(x) Q(x) dx = \frac{1}{(1/x^3)} \int \left(\frac{1}{x^3}\right)(x) dx$$

$$= x^3 \int \frac{1}{x^2}\,dx = x^3\left(-\frac{1}{x} + C\right) = Cx^3 - x^2$$

The solution is the function $y = Cx^3 - x^2$.

Example

Solve the equation $(1+x^2)dy + (y - \tan^{-1}(x))dx = 0$.

Solution

Dividing the two sides by $(1+x^2)dx$

$$\frac{dy}{dx} + \frac{y}{1+x^2} - \frac{\tan^{-1}(x)}{1+x^2} = 0$$

$$\frac{dy}{dx} + \frac{y}{1+x^2} = \frac{\tan^{-1}(x)}{1+x^2} \quad \Rightarrow \quad P(x) = \frac{1}{1+x^2}, \quad Q = \frac{\tan^{-1}(x)}{1+x^2}$$

$$\int P(x)dx = \int \frac{1}{1+x^2}\,dx = \tan^{-1}(x)$$

$$\rho(x) = e^{\tan^{-1}(x)}$$

$$e^{\tan^{-1}(x)}y = \int e^{\tan^{-1}(x)} \frac{\tan^{-1}(x)}{1+x^2}\,dx + C$$

$$z = \tan^{-1}(x) \quad \Rightarrow \quad dz = \frac{1}{1+x^2}\,dx$$

$$e^{\tan^{-1}(x)}y = \int e^z \times z\,dz + C$$

$$= ze^z - \int e^z dz + C$$

$$= ze^z - e^z + C$$

$$e^{\tan^{-1}(x)}y = \tan^{-1}(x)e^{\tan^{-1}(x)} - e^{\tan^{-1}(x)} + C$$