Sixth Order Finite Difference In Two Dimensions With Some Examples

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Abstract

A nine-points sixth – order accurate compact finite difference method for solving Laplace equation in two dimensional is present. The efficiency of the method is validated by its application to the test problems which have exact solutions. Numerical results show that this sixth order scheme has the expected accuracy. The explicit numerical solution of the parabolic equation is given, In addition, can be applicable to domain with some obstacle, as well, improve the solution of Poisson equation by taking more terms from the right hand side. All the computations are solved by the iterative method with personal computer.

Keywords: Six order finite difference, Laplace equation, parabolic equation, Poisson equation.

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(1) Introduction

The partial differential equation $\nabla u = 0$ is Laplace equation governs many important physical phenomena and its applications, numerical solution of steady state of heat transfer[10], mobile robot [2], Laplace equation using fuzzy data [9], frequency space two dimensions scalar extrapolator using finite difference operator [4], and so on. For the traditional finite difference methods, in order to increase the order of accuracy of approximations, by increasing the grid points. Dirichlet boundary conditions are used. Assume that the solution u(x, y) have sufficiently smooth and the necessary continuous partial derivatives up to certain orders. The (PDE) of Laplace equation in two dimensional can be written in the form of

$$u_{xx}(x,y) + u_{yy}(x,y) = 0, \qquad (x,y) \in \Omega$$

Where Ω is a rectangular domain, or union of rectangular domains, with suitable boundary conditions defined on $\partial\Omega$.

(2) **An Extension Method for sixth order Finite Difference in Two Dimensions**

In this section, the method is developed similar to sixth order accurate approximation [3], [7], [11]. The scheme of two dimensional rectangular domain to have nine-points stencil which is equal spaces in both directions x and y. Laplace equation given by:

$$u_{xx}(x,y) + u_{yy}(x,y) = 0 \ , \ (x,y) \in \Omega \ \text{.....................} \quad (1)$$

The central difference finite difference method is written by:

$$\delta_x^2 u_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \text{ and } \quad \delta_y^2 u_{i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j+1}}{h^2} \quad , \quad \text{where}$$

$$u_{ij} = u(x_i, y_j) \quad \text{......} \ (2)$$

By Taylor expansions at the point (x_i, y_j) , we have

$$u_{i+1,j}=u_{ij}+hu_x+\frac{h^2}{2!}u_{xx}+\frac{h^3}{3!}u_{xxx}+\frac{h^4}{4!}u_{xxxx}+\frac{h^5}{5!}u_{xxxxx}+\frac{h^6}{6!}u_{xxxxxx}+\ldots.$$

$$u_{i-1,j}=u_{ij}-hu_x+\frac{h^2}{2!}u_{xx}-\frac{h^3}{3!}u_{xxx}+\frac{h^4}{4!}u_{xxxx}-\frac{h^5}{5!}u_{xxxxx}+\frac{h^6}{6!}u_{xxxxxx}-\ldots.$$

Add the above equations we get

$$\frac{u_{i+1,j}-2u_{ij}+u_{i-1,j}}{h^2}=\frac{\partial^2 u}{\partial x^2}+\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}+\frac{h^4}{360}\frac{\partial^6 u}{\partial x^6}+\frac{h^6}{20160}\frac{\partial^8 u}{\partial x^8}+\ldots.$$

or

$$\delta_x^2 u_{i,j}=\frac{\partial^2 u}{\partial x^2}+\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}+\frac{h^6}{360}\frac{\partial^6 u}{\partial x^6}+O(h^6) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

In similar manner

$$\delta_y^2 u_{i,j}=\frac{\partial^2 u}{\partial y^2}+\frac{h^2}{12}\frac{\partial^4 u}{\partial y^4}+\frac{h^6}{360}\frac{\partial^6 u}{\partial y^6}+O(h^6) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Substitute (3) and (4) in (1) we obtain

$$\delta_x^2 u_{i,j}+\delta_y^2 u_{i,j}+T_{i,j}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Where $u_{i,j}=u(x_i,y_j)$, and

$$T_{i,j}=-\frac{h^2}{12}\left(\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}\right)_{i,j}-\frac{h^4}{360}\left(\frac{\partial^6 u}{\partial x^6}+\frac{\partial^6 u}{\partial y^6}\right)_{i,j}+O(h^6) \quad \ldots\ldots \quad (6)$$

Using the following appropriate derivatives of equation (1):

$$\frac{\partial^4 u}{\partial x^4}=-\frac{\partial^4 u}{\partial x^2\partial y^2} \text{ and } \frac{\partial^4 u}{\partial y^4}=-\frac{\partial^4 u}{\partial y^2\partial x^2} \quad \ldots\ldots\ldots\ldots\ldots \quad (7)$$

Substitute in (6) , we get:

$$T_{i,j}=-\frac{h^2}{12}\left(-2\left[\frac{\partial^4 u}{\partial x^2\partial y^2}\right]_{i,j}\right)-\frac{h^4}{360}\left(\frac{\partial^6 u}{\partial x^6}+\frac{\partial^6 u}{\partial y^6}\right)_{i,j}+O(h^6)..............(8)$$

This derivation of $O(h^6)$ scheme, the fourth order approximation of the term $\frac{\partial^4 u}{\partial x^2\partial y^2}$ in equation (8) is given by:

$$\left[\frac{\partial^4 u}{\partial x^2\partial y^2}\right]_{i,j}=\delta_x^2\delta_y^2 u_{i,j}-\frac{h^2}{12}\left(\frac{\partial^6 u}{\partial x^4\partial y^2}+\frac{\partial^6 u}{\partial x^2\partial y^4}\right)_{i,j}+O(h^4)...................(9)$$

Substituting equation (9) in (8) , we get :

$$T_{i,j}=\frac{h^2}{12}\left(2\delta_x^2\delta_y^2 u_{i,j}\right)-\frac{h^4}{360}\left(\frac{\partial^6 u}{\partial x^6}+5\frac{\partial^6 u}{\partial x^4\partial y^2}+5\frac{\partial^6 u}{\partial x^2\partial y^4}+\frac{\partial^6 u}{\partial y^6}\right)_{i,j}+O(h^6)....(10)$$

getting a compact sixth- order approximation requires compact expressions of the four derivatives of order six in (10), which can be done by further differentiating equation (1), that is :

$$\frac{\partial^6 u}{\partial x^6}+\frac{\partial^6 u}{\partial y^6}=-\left(\frac{\partial^6 u}{\partial x^4\partial y^2}+\frac{\partial^6 u}{\partial x^2\partial y^4}\right)\quad ....\quad (11)$$

Substituting equations (7) into equation (11) gives us:

$$T_{i,j}=\frac{h^2}{12}\left(2\delta_x^2\delta_y^2 u_{i,j}\right)-\frac{h^4}{360}\left(4\left[\frac{\partial^4 u}{\partial x^2\partial y^2}\right]_{i,j}\right)+O(h^6)\quad ....\quad (12)$$

The compact sixth-order approximation of the Laplace equation in two dimensional can be obtained as:

$$\frac{h^2}{6}\delta_x^2\delta_y^2 U_{i,j}+(\delta_x^2+\delta_y^2)U_{i,j}=0\quad ....\quad (13)$$

Where $U_{i,j} = u_{i,j}$ is the discrete approximation to $u_{i,j}$ satisfying the discrete formulation of equation (1) that is, $u_{i,j} = U_{i,j} + O(h^6)$

We can express the equation (13) in the form of

$$K_{20}U_{i,j} + K_{21}H + K_{22}X = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (14),$$

Where

$$H = U_{i+1,j} + U_{i-1,j} + U_{i,j} + 1 + U_{i,j-1} \quad \ldots\ldots\ldots \quad (15)$$

$$X = U_{i+1,j+1} + U_{i-1,j+1} + U_{i+1,j-1} + U_{i-1,j-1} \quad \ldots\ldots\ldots \quad (16)$$

$$K_{20} = -\frac{10}{3}, \quad K_{21} = \frac{2}{3}, \quad and \quad K_{22} = \frac{1}{6} \quad \ldots\ldots\ldots \quad (17) .$$

May write equation (15) in the form:

$$-\frac{10}{3}U_{i,j} + \frac{2}{3}\left(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1}\right)$$

or

$$+\frac{1}{6}\left(U_{i+1,j+1} + U_{i+1,j-1} + U_{i-1,j+1} + U_{i-1,j-1}\right) = 0 \quad \ldots\ldots\ldots \quad (18)$$

We can use more terms as derived in [8], we have the following correspondence formula:

$$h^2\nabla^2 = \frac{1}{6}(2H + X) - \frac{1}{420}(13H^2 + 16HX + 6X^2) + \frac{1}{630}(2H^3 + H^2 + 4HX^2) - \ldots.(19)$$

$$6h^2\nabla^2 = (2H + X) - \frac{1}{70}(13H^2 + 16HX + 6X^2) + \frac{1}{15}(2H^3 + H^2 + 4HX^2) - \ldots..(20)$$

In actual solution of Laplace equation using only one term of the right hand side of equation (20).

(3) Examples:

(1) Laplace Equation

Laplace equation it has many applications, heat transfer, potential energy, path planning for Mobile Robot [2], Laplace equation using Fuzzy Data [8], and so on …[5,6,11,12,13]. We applying the first term of equation (20) to some example and we compare the results with exact solution.

Example (1):

The Laplace equation $\Delta u = 0$, in the rectangular domain $0 \le x \le 1$, $0 \le y \le 1$, where the boundary condition and the exact solution is given by $u(x, y) = \exp(-2x)\sin(2y)$.

Solution

We solve this example by three methods one using five points diagonally, the second five points cross section, and the third one using 9-points implements equation (20).

Step length	Max error using five points diagonally	Max error using five points Cross section	Max error using nine points
0.25	8.552833E-03	3.623592326E-03	4.722111879E-07
0.2	5.782082E-03	2.581170344E-03	1.355933595E-07
0.1	1.358132E-03	6.612918926E-04	2.117365338E-09
0.05	3.375636E-04	1.676534456E-04	3.333783249E-11

(2) Parabolic Equation $U_t = C^2 \nabla^2 U$ **.**

By differentiation and substitution from the equation $U_t = C^2 \nabla^2 U$ we get the sequence [8]:

$U_{t^m} = C^{2m} \nabla^{2m} U_t$ for $m = 1, 2, \cdots$.

Then by Taylor's series

$$U(x,y,t+k)-U(x,y,t)=kU_t+\frac{k^2U_{tt}}{2!}+\frac{k^3U_{ttt}}{3!}+\cdots.$$
$$=\frac{1}{6}h^2\nabla^2U+\frac{1}{36}h^4\nabla^4U+\cdots.$$

Where we have set $k=h^2/6C^2$. Now for $h^2\nabla^2U$ we substitute the expression in terms of differences givens by equation (20), collect differences of like orders, and obtain finally:

$$U(x,y,t+k)-U(x,y,t)=\frac{1}{36}(4M+N)U(x,y,t)+\cdots.$$ where

$$M=\left(U_{i+1,j}+U_{i-1,j}+U_{i,j+1}+U_{i,j-1}\right)$$
$$N=\left(U_{i+1,j+1}+U_{i+1,j-1}+U_{i-1,j+1}+U_{i-1,j-1}\right)$$

In actual computation we drop all terms on the right in equation except the first term and represent the resulting equation by the stencil

$$U(x,y,t+k)=\frac{1}{36}\begin{bmatrix}1&4&1\\4&16&4\\1&4&1\end{bmatrix}U(x,y,t).$$

Example (2) Solve the equation $U_t=C^2\nabla^2U$ with the $U=0$ on the boundary of the square of side 6h, and the initial value of U given by $U=xy(6h-x)(6h-y)/h^4$.

K=0						
0	0	0	0	0	0	0
0	25	40	45	40	25	0
0	40	64	72	64	40	0
0	45	72	81	72	45	0
0	40	64	72	64	40	0
0	25	40	45	40	25	0
0	0	0	0	0	0	0

K=1						
0	0	0	0	0	0	0
0	21.78	35.78	40.44	35.78	21.78	0
0	35.78	58.78	66.44	58.78	35.78	0
0	40.44	66.44	75.11	66.44	40.44	0
0	35.78	58.78	66.44	58.78	35.78	0
0	21.78	35.78	40.44	35.78	21.78	0
0	0	0	0	0	0	0

K=2						
0	0	0	0	0	0	0
0	19.26	32.19	36.57	32.19	19.26	0
0	32.19	53.78	61.11	53.78	32.19	0
0	36.57	61.11	69.44	61.11	36.57	0
0	32.19	53.78	61.11	53.78	32.19	0
0	19.26	32.19	36.57	32.19	19.26	0
0	0	0	0	0	0	0

(3) Parabolic equation with obstacles

We solve the above example in (2) with some obstacles[2], in the center of the domain, we have the results as shown below.

Example (3):_ Solve the equation $U_t = C^2 \nabla^2 U$ with the $U = 0$ on the boundary of the square of side 6h, and in the center of the square, and the initial value of U in the other all points are given by $U = xy(6h - x)(6h - y)/h^4$.

K=0						
0	0	0	0	0	0	0
0	25	40	45	40	25	0
0	40	0	0	0	40	0
0	45	0	0	0	45	0
0	40	0	0	0	40	0
0	25	40	45	40	25	0
0	0	0	0	0	0	0

K=1						
0	0	0	0	0	0	0
0	21.78	35.78	40.44	35.78	21.78	0
0	35.78	0	0	0	35.78	0
0	40.44	0	0	0	40.44	0
0	35.78	0	0	0	35.78	0
0	21.78	35.78	40.44	35.78	21.78	0
0	0	0	0	0	0	0

K=2						
0	0	0	0	0	0	0
0	17.63	23.81	25.93	23.81	17.63	0
0	23.81	0	0	0	23.81	0
0	25.93	0	0	0	25.93	0
0	23.81	0	0	0	23.81	0
0	17.63	23.81	25.93	23.81	17.63	0
0	0	0	0	0	0	0

K=3						
0	0	0	0	0	0	0
0	13.13	16.08	16.81	16.08	13.13	0
0	16.08	0	0	0	16.08	0
0	16.81	0	0	0	16.81	0
0	16.08	0	0	0	16.08	0
0	13.13	16.08	16.81	16.08	13.13	0
0	0	0	0	0	0	0

(4) Poisson Equation

The sixth order finite difference method can be applied to the Poisson equation, the formula is given by the equation(20), we can use more term on the right hand side of the following equation:

$$h^2\nabla^2 = \frac{1}{6}(2H+X) - \frac{1}{420}(13H^2+16HX+6X^2) + \frac{1}{630}(2H^3+H^2+4HX^2) - ...(4.1)$$

Such that express the right hand side of (d1) by the function $f(x,y)$.

$$6h^2u(x_i,y_j) = 6h^2 f(x_i,y_j) + \left(\frac{h^4}{2}\right)\nabla^2 f(x_i,y_j) + \left(\frac{h^6}{60}\right)\nabla^4 f(x_i,y_j) + \frac{h^6}{15} f_{xxyy}(x_i,y_j)...(4.2)$$

Example (4.1)

Applied Poisson equation in a rectangular domain $0 \le x \le 1$, and $0 \le y \le 1$. Where the boundary condition and the exact solution is given by the equation $u(x,y) = \sin(\pi x)\sin(\pi y)$. Solution: we take multi step length to divided the square domain and we take the first term on the hand side of (4.2), obtained the results shown in table-1-. As well if we take two terms from the right hand side obtained the results shown in tabl-2-. By comparing the approximate solutions with the exact, we have the following results

Table-1-

Step length h	Max absolute error using one term by 9- points
0.25	1.07215197317066E-01
0.2	6.11825763467646E-02
0.1	1.657949043637963E-02
0.05	4.124499304353879E-03

Table-2-

Step length h	Max absolute error using two terms by 9- points
0.25	6.759250669847505E-03
0.2	2.450206780333142E-03
0.1	1.643228010298481E-04
0.05	1.019366904331243E-05

If we take more terms from the right hand side the solution is very significant as it is clear in the difference between two results in the two tables.

Remark: All the computations of the above results are in personal computer (Laptop) using FORTRAN language.

References

[1] Ali A. Mhassin, Poisson Equation in Irrectangular Domain, Faculty Of Science Bulletin, vol. 16 Oct. (2003), 49-55.

[2] Azali Saudi and Jumat Sultaiman, Path Planning for Mobile Robot using 4EGSOR via Nine point Laplacian (4EGSOR9L) Iterative Method International Journal of Computer Applications, vol.53 No.16, Sept (2102).

[3] Bouthina S. Ahmed "Multigrid Method for Solving two dimensional Poisson Equation with Sixth Order Finite Difference Method", international Mathematical Forum, Vol. 6, (2011), no.55,2729-2736.

[4] Changsoo Shin, Heejeung Shon, A frequency space 2D scalar wave extrapolator using extended 25 points finite difference operator, Geophysics vol, 63, No.1, (1998), P.289-296.

[5] F. Ihlenburg I. Babuska, Finite element solution to Helmholtz equation with high wave number part 1: the h-version of the FEM, Computer and

Mathematics with Applications 30 (1995) 9-37.

[6] G. Sutmann, Compact finite difference schemes of sixth order for the Helmholtz equation, Journal of Computational and applied mathematics 203(2007) 15-31.

[7] Majid Nabavi "A new 9-piont sixth –order accurate compact finite-difference method for Helmholtz equation", Journal of Sound and Vibration 307(2007)972-982.

[8] Milne E. W. "Numerical Solution of Differential Equations", Dover Publications, Inc., New York, (1970).

[9] Raphel Kumar Saikia, Numerical Solution of Laplace Fuzzy Data,

International of Computer Science, Vol. 38 No. 8, January (2012).

[10] Victor ANJO, Numerical Simulation of Steady State Conduction Heat Transfer during the Solidification of Aluminum Casting in Green Sand Mould. Leonardo Electronic Journal of Practices and Technologies, Issue 20, January –June (2012), p. 15-24.

[11] W.L. Briggs, V. E. Henson, S.E. McCormick, A multigrid Tutorial, Second ed. SIAM, Philadelphia, PA, (2000).

[12] Y. Wang, J. Zhang sixth order compact schame combined with

Multigrid method and extrapolation technique for 2D Poisson equation J. Computation. Phys., 228 (2009), 137-146.

[13] Yin Wang, Jun Zhang, Sixth order compact scheme combined with multigrid method and extrapolation technique for 2D Poisson equation, Journal of Computational Physics, 228(2009) 137-146.

الفروق المقسومة من الرتبة السادسة في بعدين مع أمثلة

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المستخلص

الفروق المقسومة بتسعة نقاط من الرتبة السادسة استعملت لحل معادلة لابلاس في متغيرين. كفاءة هذه الطريقة واضحة من خلال تطبيقها ومقارنتها مع الحل المضبوط. الحل الصريح لمعادلة القطع المكافئ إضافة إلى تطبيقها على نطاق يحتوي على مناطق معزولة, وكذلك يمكن تطبيقها على معادلة باسون مع إمكانية زيادة دقتها بإضافة عدد أكثر من الحدود من الطرف الأيمن. والنتائج العددية لهذه الطريقة مبينة في الأمثلة التي توضح ذلك.

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