Al-Mansour University College

قسم الهندسة المدنية

Civil Eng. Dept.

المرحلة الاولى

1st. Stage

Mathematics I

2022- 2023

1

الرياضيات

Esraa Thamir

Notations & Abbrevations مــــلاحظـــات واخـــتصـــارات

α =Alpha, β =Beta ɣorΓ=Gamma $\delta or \Delta$= delta

θ =Theta λ =lemda ξ= Eata ζ=zeeta

μ=Mou $\sigma or \sum$ =Segma $\pi or \prod$ =pi $\emptyset or \Phi$= Fi

$\psi or \Psi$=Epsi ϵ ≡Epsilent τ =tow ρ =row

∇=caral $\omega or \Omega$=omega

1st ≡ First, 2nd ≡ Second, 3rd ≡Third, 4th =Fourth,

no. ≡Number no.′s ≡Numbers, +ive≡positive , - ive= negative

,

∋≡ such that, ∀≡ for each, ∃=Thereexist

w.r.t= with respect to, Lim= limit,

R≡ Range, Int≡ Intercept,

Asy. ≡Asymptote, V. ≡vertical, H. ≡ Horizontal,

R≡ Set of real numbers= $\{x: -\infty < x < \infty\}$

₵= set of complex numbers

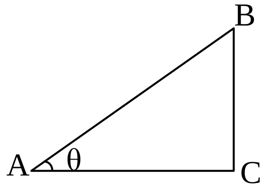
=Equal ≡ Indentical ≥ Greater than or equal

≤ less than or equal ⇒ Implies → Approach

Some Trigonometric Identities

$\sin\theta = \frac{BC}{AB}$, $\cos\theta = \frac{AC}{AB}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{BC}{AC}$$



$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{AC}{BC}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{AB}{AC} \quad , \csc\theta = \frac{1}{\sin\theta} = \frac{AB}{BC}$$

$-1 \le \sin\theta \le 1 \quad and \quad -1 \le \cos\theta \le 1$

$-\infty \le \tan\theta \le \infty \quad and \quad -\infty \le \cot\theta \le \infty$= -

$\{\sec\theta \le -1 \ or \sec\theta \ge 1\} \quad and \ \{\csc\theta \le -1 \ or \csc\theta \ge 1\}$

$\sin(\theta 1 \pm \theta 2) = \sin\theta 1 \cos\theta 2 \pm \sin\theta 2 \cos\theta 1$

$\cos(\theta 1 \pm \theta 2) = \cos\theta 1 \cos\theta 2 \mp \sin\theta 1 \sin\theta 2$

$$\tan(\theta 1 \pm \theta 2) = \frac{\tan\theta 1 \pm \tan\theta 2}{1 \mp \tan\theta 1 \tan\theta 2}$$

$\sin(2\theta) = 2\sin\theta\cos\theta$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}, \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

The solution of $ax^2 + bx + c = 0 \quad is \quad x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

The Indeterminate forms

$\frac{0}{0}, \frac{\infty}{\infty}, 0^{\circ}, 1^{\infty}, \infty^{\circ}, \infty - \infty, 0.\infty$

Equation of a straight line معادلة الخط المستقيم

The eq. of a st. line is **ax+by+c=0**

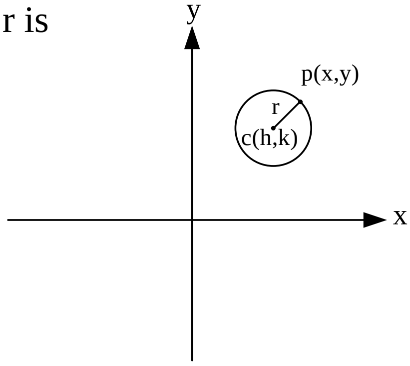
Where a, b, c are constants

Circle: Is the locus of all points in plane whose distance from fixed point is constant.

The fixed point is called the center of the circle and denoted by c (h,k) and the constant distance is called the radius of the circle and denoted by r.

The eq. of the circle with center at (h,k) and radius r is

$r^2 = (x-h)^2 + (y-k)^2$ _______(1)



Note: If h=k-0, then eq. (1) becomes

$r^2 = x^2 + y^2$

Inequalities المتراجحات

If a and b ate real no's, then one of the following is true: a>b or a=b or a<b

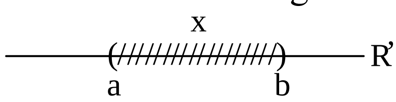
Notes:

1. If a>b then –a< -b
2. If a>b then $\frac{1}{a} < \frac{1}{b}$

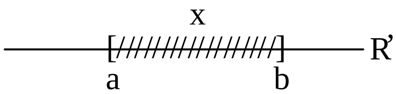
Intervals الـــفـــتـــرات

Defn. An interval is a set of no's x having one of the following from:

(i) Open interval: a<x<b ≡ (a, b)



(ii)

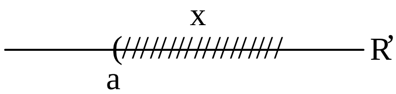


(iii) Half open from the left or half close from the right: a<x≤b (a,b]

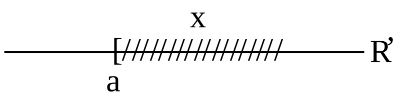
(iv) Half close from the left or half open from the right: a≤ x<b [a,b)

Notes:

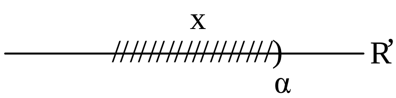
(1) a< x<∞ ≡ a< ≡ (a, ∞)



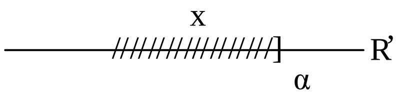
(2) a≤ x< ∞ ≡ a≤ x ≡ [a, ∞)



(3) ∞< x≤ a ≡ x<a ≡ (-∞, a)



(4) ∞< x≤ a ≡ x≤ a ≡ (-∞, a]



Absolute Value الــقيمـــة المـطـلـقـة

Defn: The absolute value of a real no. x is define as

|x| = $\begin{cases} x\ ,\ x \geq 0 \\ -x\ \ \ ,x < 0 \end{cases}$, 1×1=$\sqrt{x^2}$

Properties of absolute values خصــائص القيـــم المـطلقـــة

1. $|x.y| = |x|.|y|\ and\ = \frac{|x|}{|y|}$
2. $|-x| = |x|$
3. $|x+y| \leq |x| + |y|$
4. $|x| < a\ \ mean\ \ \ \ \ -a < x < a$
5. $|x| \leq a\ \ mean\ \ \ \ \ -a \leq x \leq a$
6. $|x| > a\ \ mean\ \ \ \ \ x < -a\ \ or\ \ x > a$
7. $|x| \geq a\ \ mean\ \ \ \ \ x \leq -a\ \ or\ \ x \geq a$

Example: Find the solution set of the following ineq's:

$(1)\left|\frac{3x+1}{2}\right| < 1$, $(2)|x+1| \geq 5$

Solu.

$(1)\left|\frac{3x+1}{2}\right| < 1 \Rightarrow -1 < \frac{3x+1}{2} < 1 \Rightarrow -2 < 3x+1 < 2$

$\Rightarrow -3 < 3x < 1 \quad \Rightarrow -1 < x < \frac{1}{3}$

$x \in (-1, \frac{1}{3})$

2. $|x+1| \geq 5 \Rightarrow \quad x-1 \leq -5\ \ or\ x-1 \geq 5 \Rightarrow x \leq -4$

Or $x \geq 6$

Graphs and Functions:

Defn. : The <u>solution</u> set or <u>locus</u> of an equation in two unknow consists of all points in the plane whose coordinates satisfy the eq. A geometrical representation of the locus is called the graph of the equation.

Ex. Sketch the graph of the following eq's.:

1. $2x + 3y = 6$

2. y= $\begin{cases} x & 0 \le x \le 1 \\ 2 - x & 1 \le x \le 2 \end{cases}$

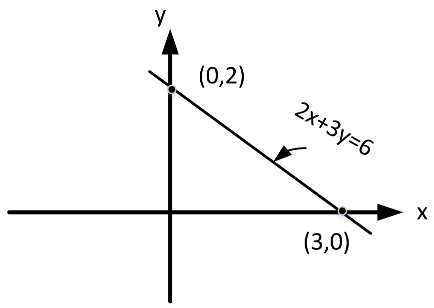
3. y= $\begin{cases} -x & , x < 0 \\ x^2 & , 0 \le x \le 1 \\ 1 & , 1 < x \end{cases}$

4. $y = |x^2 - 1|$

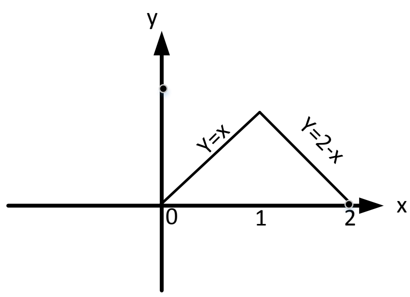
5. $16x^2 + 25y^2 = 400$

Solu.

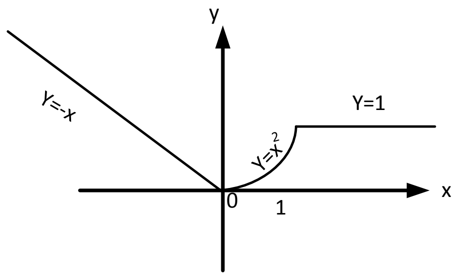
1. $2x + 3y = 6$



2. $y = \begin{cases} x & , 0 \le x \le 1 \\ 2 - x & , 1 < x \le 2 \end{cases}$



3. y= $\begin{cases} -x & , x < 0 \\ x^2 & , 0 \le x \le 1 \\ 1 & , 1 < x < 3 \end{cases}$



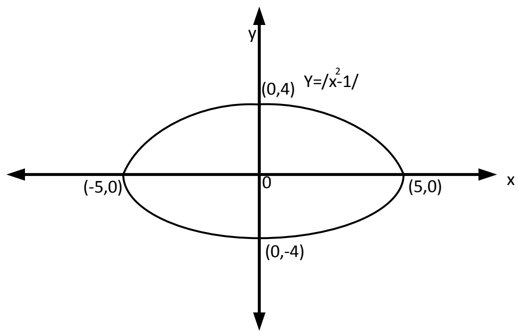
4. $y = |x^2 - 1| = \begin{cases} x^2 - 1 & , \quad x^2 - 1 \ge 0 \\ -(x^2 - 1), & x^2 - 1 < 0 \end{cases}$

$$= \begin{cases} x^2 - 1 & , \quad (x - 1)\,(x + 1) \ge 0 \\ 1 - x^2, & \quad (x - 1)\,(x + 1) < 0 \end{cases}$$

$$= \begin{cases} x^2 - 1 & , \quad x \le -1 \quad or \quad x \ge 1 \\ 1 - x^2 & , \quad -1 < x < 1 \end{cases}$$

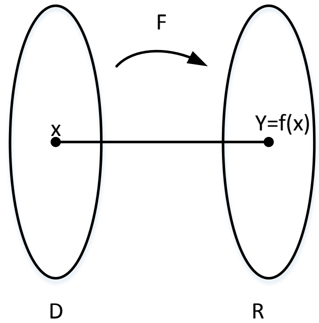
5. $16x^2 + 25y^2 = 400 \Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1,$

ellipse



<u>Defn. (Function):</u> A function f from a set D to a set R is a rule that assigns a single element y$\in R$ to each element x $\in D$

Note: The element y$\in R$ denoted by f(x), the set D is called the domain of f, and the set R is called the range of f.



Defn. $f(x)$ is an even function if $f(-x) = f(x)$

$f(x)$ is an odd function if $f(-x) = -f(x)$

Ex.

1. $f(x) = x^2 cosx \Rightarrow f(-x) = (-x)^2 \cos(-x)$

$$= x^2 \text{cosx} = \text{f(x)}$$

$\therefore f(x)$ is an even function

2. $f(x) = \frac{x^2-1}{sinx} \Longrightarrow f(-x) = \frac{(-x^2)-1}{\sin(-x)} = \frac{x^2-1}{-sinx} = -f(x)$

$\therefore f(x)$ is an odd function

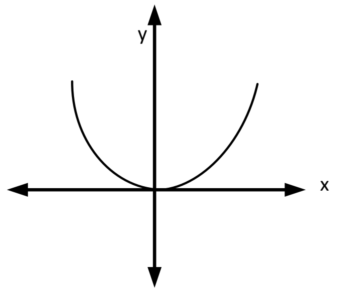
Note: we may define

The domain D is the set of all values of x for which y is define

The range R is the set of all values of y for which x is define

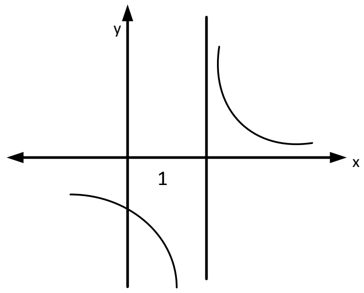
Ex. Find the domain and the range of the following functions:

1. $y = f(x) = x^2, D: all\ x\ ,\ \ R: y \geq 0$

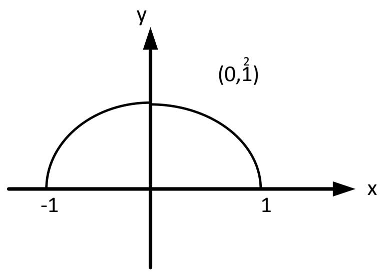


2. $y = \frac{1}{x-1}, D: x \neq 1$

$$x = \frac{y+1}{y}, R: y \neq 0$$



3. $y = \sqrt{4-x^2},$ $D: -2 \leq x \leq 2$ $R: 0 \leq y \leq 2$



4. $y = f(x) = \sqrt{x^2 - 4x + 3}$

$x^2 - 4x + 3 \geq 0 \Rightarrow D: x \leq 1\ or\ x \geq 3$

$y^2 = x^2 - 4x + 3 \qquad \Rightarrow x^2 - 4x + 3 - y^2 = 0$

$$x = \frac{4 \pm \sqrt{16 - 4(3 - y^2)}}{2} = \frac{4 \pm \sqrt{4 + 4y^2}}{2} = 2 \pm \sqrt{1 + y^2}$$

$\therefore R: ally$

5. $y = \sqrt{2 - \sqrt{x}}$

For $\sqrt{x}$ it must be $x \geq 0$

$2 - \sqrt{x} \geq 0 \Rightarrow 2 \geq \sqrt{x} \Rightarrow 4 \geq x$

$\therefore D: 0 \leq x \leq 4$

$x = (2 - y^2)^2, R: ally$

Intercepts, Symmetry, and Asymptotes

1. To find x-intercepts, set y=0 and solve for y.

To find y-intercepts, set x=0 and solve for x.

2. The locus is symmetric w.r.t. the

i. x-axis (x,-y) ⇔ (x, y)

ii. y-axis (-x, y) ⇔ (x, y)

iii. origin (-x,-y) ⇔ (x, y)

3. (i) A line x=a near which a locus goes of f to ∞ is called V. Asy.

(ii) A line y=b near which a locus goes of f to ∞ is called H. Asy.

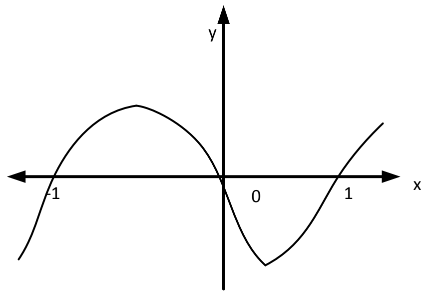
Ex. Find the domain, the range, intercepts, symmetry, and asymptotes if they exist for the following functions Sketch.

1. y=f(x)=$x^3 - x$, D: all x , R: all y

(0,0), (1,0), (-1,0) are x-intercepts

(0,0) is y-intercept

Symmetric w.r.t. origin only No asymptotes.



2. y=f(x)=$\frac{1}{x^2-1}$, D: x $\neq \pm 1$

$$x = \mp\sqrt{\frac{y+1}{y}},$$

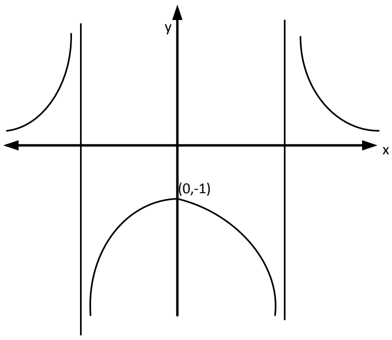
$$R: y > 0 \quad or\ y \leq -1$$

(0,-1) is y-intercepts

Symm. W.r.t. y-axis only

x=$\pm$1, V.Asy.

y=0, H.Asy.



Limit & Continuityالـــغـــايـــة والاستمـــراريـة

Notation

When f(x) lends to the number L as x tends to the number a we write f(x) →L as x→ 0

or $\lim_{x\to a} f(x) = L$

Ex.(1) Let f(x)=2x+5

Evaluate f(x) at x=1.1 , 1.01, 1.001, 1.0001,……

F(1.1)= 2(1.1)+5=7.2

F(1.01)= 2(1.01)+5=7.02

F(1.001)=2(1.001)+5=7.002

F(1.0001)=2(1.0001)+5=7.0002

We see that f(x)→7 as x→1

Or

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}(2x+5) = 2(1)+5 = 7$$

Ex. (2) If $f(x) = \frac{x^2-3x+2}{x-2}, \quad x \neq 2$ find $\lim_{x\to 2} f(x)$.

$\lim_{x\to 2} f(x) = \lim_{x\to 2}\frac{x^2-3x+2}{x-2} = \frac{4-6+2}{2-2} = \frac{0}{0}$ meaning less

$\lim_{x\to 2} f(x) = \lim_{x\to 2}\frac{x^2-3x+2}{x-2} = \frac{(x-2)(x-1)}{(x-2)} = \lim_{x\to 2}(x-1) = 2-1 = 1$.

Ex. (3) Evaluate the following limits, if they exist

1. L= $\lim_{x\to -1}\frac{\sqrt{2+x}-1}{x+1}, x \neq -1, x \geq -2$

L= $\lim_{x\to -1}\frac{\sqrt{2+x}-1}{x+1}.\frac{\sqrt{2+x}+1}{\sqrt{2+x}+1} = \lim_{x\to -1}\frac{(2+x-1)}{(x+1)\sqrt{2+x}+1}$

$$= \frac{1}{\sqrt{2-1}+1} = \frac{1}{1+1} = \frac{1}{2}$$

2. $\lim_{x\to 2}\frac{2-x}{2-\sqrt{2x}}, x \neq 2, x \geq 0$

$$L = \lim_{x\to 2}\frac{2-x}{2-\sqrt{2x}}.\frac{2+\sqrt{2x}}{2+\sqrt{2x}} = \lim_{x\to 2}\frac{(2-x)(2+\sqrt{2x})}{4-2x}$$

$$= \lim_{x \to 2} \frac{(2-x)(2+\sqrt{2x})}{2(2-x)} = \frac{2+\sqrt{4}}{2} = \frac{2+2}{2} = 2$$

H.w

3. $\lim_{x \to 3} \frac{\sqrt{3x}-3}{x-3}, \quad x \neq 3$

4. $\lim_{x \to 2} \frac{x^4-2x^2-8}{x^2-4}, \quad x \neq 2$

5. $\lim_{x \to a} \frac{\sqrt{x^2+1}-\sqrt{a^2+1}}{x-a}, \quad x \neq a$

6. $\lim_{x \to 0} \frac{1}{x}\left(\frac{1}{x-2} - \frac{1}{2}\right), \quad x \neq 0{,}2$

7. $\lim_{x \to 0} \frac{(1+x)^{3/2}-1}{x}, \quad x \neq 0$

Theorems on Limits (Calculation Technique)

1. Uniqueness of limit **وحدانية الغاية**

If $\lim_{x \to a} f(x) = L_1$, and$\lim_{x \to a} f(x) = L_2 \Longrightarrow L_1 = L_2$

2. Limit of constant

If f(x)=c, c is a constant then $\lim_{x \to a} f(x) = \lim_{x \to a} c = c$.

3. Obvious limit

If f(x)=x then $\lim_{x \to a} f(x) = \lim_{x \to a} x = a$

4. Limit of sum

If f(x)=f_1(x)$\pm$ f_2(x) $\pm$ f_3(x) $\pm$…$\pm$ f_n(x) and $\lim_{x \to a} fl(x) = L_i$

$i = 1,2,\dots,n$ then

$$\lim_{x \to a} f(x) = \lim_{x \to a}[f_1(x) \pm f_2(x) \pm \dots . \pm f_n(x)]$$

$$= \lim_{x \to a} f_1(x) \pm \lim_{x \to a} f_2(x) \pm \dots . \pm \lim_{x \to a} f_n(x)$$

$$= L_1 \pm L_2 \pm \cdots . \pm L_n = \sum_{i=1}^{n} L_i$$

5. Limit of a product

If f(x) =f_1(x). f_2(x) . …. f_n(x) and $\lim_{x \to a} f_i(x) = L_i$

$i = 1,2,\dots,n$ then $\lim_{x \to a} f(x) = \lim_{x \to a}[f_1(x). f_2(x). \dots . . f_n(x)]$

$\lim_{x \to a} f(x) = \lim_{x \to a} f_1(x). \lim_{x \to a} f_2(x)$ ....$\lim_{x \to a} f_n(x)$.

$$= L_1 . L_2 . \ldots L_n = \prod_{i=1}^{n} L_i$$

6. Limit of a Quotient

If $f(x) = \frac{g(x)}{h(x)}$ and $\lim_{x\to a} g(x) = L_1$, and $\lim_{x\to a} h(x) = L_2 \neq 0$,

then $\lim_{x\to a} f(x) = \lim_{x\to a} \frac{g(x)}{h(x)} = \frac{\lim_{x\to a} g(x)}{\lim_{x\to a} h(x)} = \frac{L_1}{L_2}$

Ex. (4) Evaluate the following limits

i. $\lim_{x\to 1} \frac{x^3-1}{x-1}, x \neq 1$

$$\lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x-1)} = (1^2+1+1) = 3$$

ii. $\lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right), h \neq 0$

$$\lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right) = \lim_{h\to 0} \frac{1}{h} \frac{x-x-h}{(x+h)x} = \lim_{h\to 0} \frac{-h}{h(x+h)x}$$

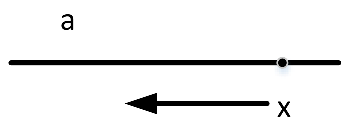
$$= -\lim_{h\to 0} \frac{1}{x(x+h)} = -\frac{1}{x(x+0)} = \frac{-1}{x^2}$$

iii. $\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}, \quad h \neq 0$ **(H.W)**

One sided and Two Sided limits (Right limits and left limits)

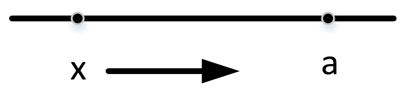
Sometimes the values of a function $f(x)$tend to different limits as x tends a from different sides. When this happens we the limit of $f(x)$as x approaches a from the right by the right-hand limit and denoted by

$$\lim_{x\to a^+} f(x) = L$$



And the limit of $f(x)$as x approaches a from the left by the left- hand limit and denoted by

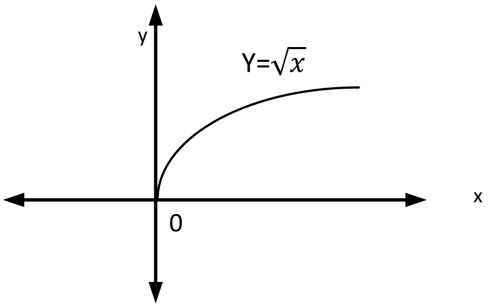
$$\lim_{x\to a^-} f(x) = L$$



Note from uniqueness theorem of the limit, we know that if limit exist then it is a unique, so that

$\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$

Ex.(5) $f(x) = \sqrt{x}, D: x \geq 0.$ Find $\lim_{x\to 0} f(x) = ?$



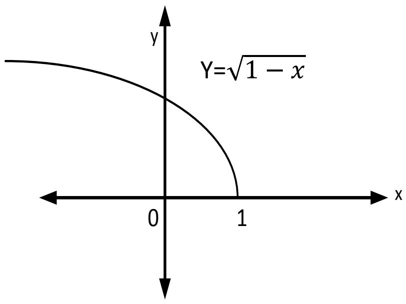
Since $\sqrt{x}$ is not define for –ive Value of x, so we restrict to +ive Value of x

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sqrt{x} = \sqrt{0} = 0$

$= \lim_{x\to 0} f(x)$

(This example of one-sided limit)

Ex.(6) $f(x) = \sqrt{1-x}, D: x \leq -1.$ Find $\lim_{x\to 1} f(x) = ?$



Since $\sqrt{1-x}$ is not define for x$>$ 1, so we restrict to values of $x \leq 1$

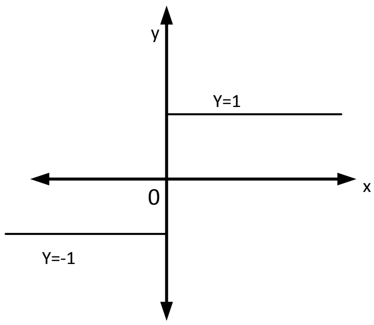
$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt{1-x} = \sqrt{1-1} = \sqrt{0}$

$= 0$

$= \lim_{x\to 1} f(x)$

(This example of one –sided limit)

Ex.(7) $f(x) = \frac{x}{|x|}$. Find $\lim_{x\to 0} f(x) = ?$



$since |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

$\therefore f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 1 = 1$ and $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-1) = -1$

Since $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$, the $\lim_{x\to 0} f(x)$ does not exist (This example of two-sided limit)

H.W

Ex. (8) $f(x) = \frac{x\sqrt{x^2+1}}{|x|}, x \neq 0$ Find $\lim_{x\to 0^+} f(x), \lim_{x\to 0^-} f(x) and \lim_{x\to 0} f(x)$

Ex. (9) $f(x) = \frac{\sqrt{4-x^2}}{\sqrt{6-5x+x^2}}$ What is the domain $\lim_{x\to 2^-} f(x), \lim_{x\to 2^+} f(x), \lim_{x\to 2} f(x)$.

Soul.

$D: -2 \leq x < 2$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{\sqrt{4-x^2}}{\sqrt{6-5x+x^2}} = \lim_{x\to 2^-} \frac{\sqrt{2-x}\sqrt{2+x}}{\sqrt{2-x}\sqrt{3-x}}$$

$$= \lim_{x\to 2^-} \frac{\sqrt{2+x}}{\sqrt{2+x}} = \frac{\sqrt{2+2}}{\sqrt{3-2}} = \sqrt{4} = 2$$

$\lim_{x\to 2^+} f(x)$ is not define, so $\lim_{x\to 2} f(x) = \lim_{x\to 2^-} f(x) = 2$

Ex.(10) $f(x) = |x-1|$. Find $\lim_{x\to 1^+} f(x), \lim_{x\to 1^-} f(x), and \lim_{x\to 1} f(x)$

Soul.

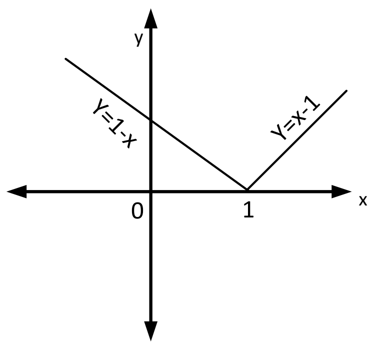
$$f(x) = |x-1| = \begin{cases} (x-1), & x-1 \geq 0 \\ -(x-1), & x-1 < 0 \end{cases}$$

$$= \begin{cases} x-1, & x-1 \geq 0 \to x \geq 1 \\ 1-x, & x-1 < 0 \to x < 1 \end{cases}$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-1) = 1-1 = 0$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (1-x) = 1-1 = 0$$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1^+} |x-1| = 0$$



Limits at Infinity

We note that when the limit of a function f(x) exist as x approaches infinity, we write $\lim_{x\to\infty} f(x) = L$.

So, we write

$\lim_{x\to\infty} f(x) = L$ for +ive values of x and $\lim_{x\to-\infty} f(x) = L$ for –ive values of x

For one-sided and two sided limits, we have $\lim_{x\to\infty} f(x) = L$ if and only if $\lim_{x\to+\infty} f(x) = L$ and $\lim_{x\to-\infty} f(x) = L$.

Some obvious limits

1. **If k is constant, then** $\lim_{x\to+\infty} k = k$ and $\lim_{x\to-\infty} k = k$.
2. $\lim_{x\to\infty} \frac{1}{x} = 0$, $\lim_{x\to+\infty} \frac{1}{x} = 0$, $\lim_{x\to-\infty} \frac{1}{x} = 0$
3. $\lim_{x\to 0} \frac{1}{x} = \infty$, $\lim_{x\to 0^+} \frac{1}{x} = +\infty$, $\lim_{x\to 0^-} \frac{1}{x} = -\infty$

Ex. (11) Find the following limits:

1. $\lim_{x\to\infty} \frac{x}{2x+3} = \lim_{x\to\infty} \frac{1}{2+\frac{3}{x}} = \frac{1}{2+0} = \frac{1}{2}$.

2. $\lim_{x\to\infty} \frac{2x^2+3x+5}{5x^2-4x+1} = \lim_{x\to\infty} \frac{2+\frac{3}{x}+\frac{5}{x^2}}{5-\frac{4}{x}+\frac{1}{x^2}} = \frac{2+0+0}{5-0-0} = \frac{2}{5}$.

3. $\lim_{x\to\infty} \frac{2x^2+1}{3x^3-2x^2+5x-2} = \lim_{x\to\infty} \frac{\frac{2}{x}+\frac{1}{x^3}}{3-\frac{2}{x}+\frac{5}{x^2}-\frac{2}{x^3}} = \frac{0+0}{3-0+0-0} = 0$.

4. $\lim_{x\to\infty} \frac{2x^3+2x-1}{x^2-2x+2} = \lim_{x\to\infty} \frac{2+\frac{2}{x^2}-\frac{1}{x^3}}{\frac{1}{x}-\frac{5}{x^2}-\frac{2}{x^3}} = \frac{2+0-0}{0-0+0} = \infty$.

That is the limit does not exist

5. $lim_{x\to\infty} \sqrt{x} = \lim_{x\to+\infty} \sqrt{x} = +\infty or \infty$.

6. $lim_{x\to\infty} \left(2 + \frac{sinx}{x}\right) = lim_{x\to\infty} 2 + lim_{x\to\infty} \frac{sinx}{x}$,but $lim_{x\to\infty} 2 = 2$ and $lim_{x\to\infty} \frac{sinx}{x} = 0$

Because -1$\leq sin \leq 1$, $then\ lim_{x\to\infty} \left(2 + \frac{sinx}{x}\right) = 2 + 0 = 2$.

7. $\lim\limits_{x\to-\infty}\left(2x+\frac{3}{x}\right)=-\infty+0=-\infty$.

8. $\lim\limits_{x\to2^-}\left(\frac{1}{x^2-4}\right)=\frac{1}{0}=-\infty$ and $\lim\limits_{x\to2^+}\left(\frac{1}{x^2-4}\right)=\frac{1}{0}=+\infty$.

9. $\lim\limits_{x\to\infty}(\sqrt{x^2+1}-x)=\infty-\infty$(meaning less)

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right).\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}=\lim_{x\to\infty}\frac{x^2+1-x^2}{\sqrt{x^2+1}+x}$$

$$=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}.$$

$$=\lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x^2}}+1}=\frac{0}{\sqrt{1+0}+1}=\frac{0}{2}=0\ .$$

10. $\lim\limits_{x\to\infty}\left(\sqrt{x^2+2x}-x\right)=\lim\limits_{x\to\infty}\left(\sqrt{x^2+2x}-x\right).\frac{\sqrt{x^2+2x}+x}{\sqrt{x^2+2x}+x}$

$$=\lim_{x\to\infty}\frac{2x}{\left(\sqrt{x^2+2x}+x\right)}=\lim_{x\to\infty}\frac{2}{\sqrt{1+\frac{2}{x}}+1}=\lim_{x\to\infty}\frac{2}{\sqrt{1+0}+1}=\frac{2}{2}=1\ .$$

More About Asymptotes

Given y=f(x). A line y=mx+b is an asymptote for f(x)

(1) $m=\lim\limits_{x\to\infty}\frac{f(x)}{x}$ **(2)** $b=\lim\limits_{x\to\infty}(f(x)-mx)$

Ex. (12) Find the asymptotes of the following functions:

1. $y=f(x)=x+\frac{1}{x}=\frac{x^2+1}{x}$

$x=0$ is V. Asy .

$x^2+1=yx\Rightarrow\quad x^2-yx+1=0\Rightarrow\quad x=\dfrac{y\pm\sqrt{y^2-4}}{2}$

No. H.Asy

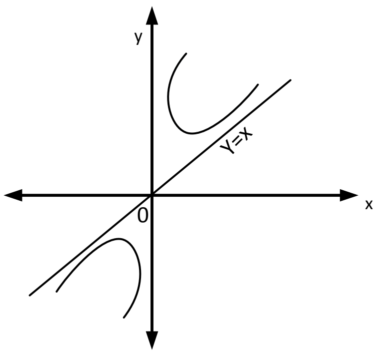
Let y=mx+b be an asy .

$$m=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{\frac{x^2+1}{x}}{x}=\lim_{x\to\infty}\frac{x^2+1}{x^2}=1$$

$$b=\lim_{x\to\infty}(f(x)-mx)=\lim_{x\to\infty}\left(\frac{x^2+1}{x}-x\right)$$

$$=\lim_{x\to\infty}\frac{x^2+1-x^2}{x}=\lim_{x\to\infty}\frac{1}{x}=0$$

$\therefore y=x$ is an asy .



2. $y=f(x)==\frac{x^2-3}{2x-4}, \quad x=2\ is\ V.Asy$.

$$x^2-3=2yx-4y \Longrightarrow x^2-2yx+4y-3=0$$

$$x=\frac{4y\pm\sqrt{4y^2-4(4y-3)}}{2}=\frac{4y\pm\sqrt{4y^2-16y+12}}{2}=\frac{4y\pm2\sqrt{y^2-4y+3}}{2}$$

$x=2y\pm\sqrt{y^2-4y+3}$ No. H. Asy .

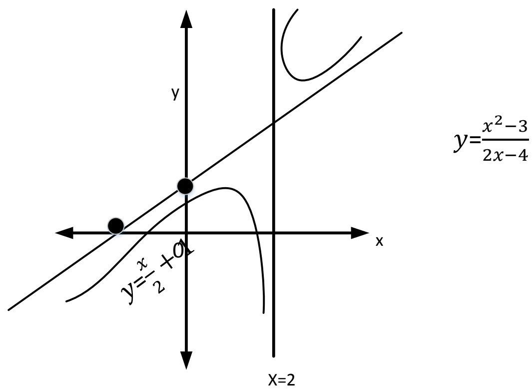
Let y=mx+b be an asymptote .

$$m=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{\frac{x^2-3}{2x-4}}{x}=\lim_{x\to\infty}\frac{x^2-3}{2x^2-4x}=\lim_{x\to\infty}\frac{1-\frac{3}{x^2}}{2-\frac{4}{x}}=\frac{1-0}{2-0}=\frac{1}{2}$$

$$b=\lim_{x\to\infty}(f(x)-mx)=\lim_{x\to\infty}\left(\frac{x^2-3}{2x-4}-\frac{1}{2}x\right)=\lim_{x\to\infty}\frac{x^2-3-x^2+2x}{2(x-2)}$$

$$=\lim_{x\to\infty}=\frac{2x-3}{2x-4}=\lim_{x\to\infty}=\frac{2-\frac{3}{x}}{2-\frac{4}{x}}=\frac{2-0}{2-0}=\frac{2}{2}=1$$

$\therefore y=\frac{x}{2}+1$ is an asymptote



<u>Sandwich Theorem</u> If $g(x) \le f(x) \le h(x)$ and if $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$ then $\lim_{x\to a} f(x) = L$

<u>Ex.(13)</u>

Find $\lim_{x\to\infty} f(x) if \ \frac{2x+3}{x} \le f(x) \le \frac{2x^2+5x}{x^2}$

$$\lim_{x\to\infty} \frac{2x+3}{x} = \lim_{x\to\infty}\left(2+\frac{3}{x}\right) = 2+0 = 2$$

$$\lim_{x\to\infty} \frac{2x^2+5x}{x^2} = \lim_{x\to\infty}\left(2+\frac{5}{x}\right) = 2+0 = 2$$

∴ By Sandwich theorem $\lim_{x\to\infty} f(x) = 2$.

Theorem (1) If θ is a measured in radian, then

$$\lim_{\theta\to 0}\frac{sin\theta}{\theta} = 1$$

Theorem (2) $\lim_{\theta\to 0}\frac{1-cos\theta}{\theta} = 0$

<u>Proof:</u> $\lim_{\theta\to 0}\frac{1-cos\theta}{\theta}.\frac{1+cos\theta}{1+cos\theta} = \lim_{\theta\to 0}\frac{1-cos^2\theta}{\theta(1+cos\theta)}$

$$\lim_{\theta\to 0}\frac{sin^2\theta}{\theta(1+cos\theta)} = \lim_{\theta\to 0}\frac{sin\theta}{\theta}.\lim_{\theta\to 0}\frac{sin\theta}{1+cos\theta}$$

$= (1)\frac{\sin(0)}{1+\cos(0)} = (1).\frac{0}{1+1} = \frac{0}{2} = 0$.

Ex.(14) Find the following Limits:

a. $\lim_{x\to 0}\frac{sin3x}{x} = 3\lim_{3x\to 0}\frac{sin3x}{3x} = 3(1) = 3$.

$as\ x \to 0 \Longrightarrow 3x \longrightarrow 0$

b. $\lim_{x\to 0}\frac{sin5x}{sin3x} = \lim_{x\to 0}\frac{\frac{sin5x}{x}}{\frac{sin3x}{x}} = \frac{\lim_{x\to 0}\frac{sin5x}{x}}{\lim_{x\to 0}\frac{sin3x}{x}}$

As $x \to 0 \Longrightarrow$

And $3x \to 0, 5x \to 0$

$$\lim_{x\to 0}\frac{sin5x}{sin3x} == \frac{\lim_{5x\to 0}\frac{sin5x}{5x}}{\lim_{3x\to 0}\frac{sin3x}{3x}} = \frac{5(1)}{3(1)} = \frac{5}{3}.$$

c. $\lim_{x\to\frac{\pi}{2}} \frac{cosx}{x-\frac{\pi}{2}} = \lim_{x\to\frac{\pi}{2}} \frac{\sin(\frac{\pi}{2}-x)}{x-\frac{\pi}{2}} = \lim_{x\to\frac{\pi}{2}} \frac{-\sin(x-\frac{\pi}{2})}{x-\frac{\pi}{2}}$

As $x \to \frac{\pi}{2} \Longrightarrow x - \frac{\pi}{2} \longrightarrow 0$

$$\lim_{x\to\frac{\pi}{2}} \frac{cosx}{x-\frac{\pi}{2}} = -\lim_{x-\frac{\pi}{2}\longrightarrow 0} \frac{\sin\left(\frac{\pi}{2}-x\right)}{x-\frac{\pi}{2}} = -1\,.$$

d. $\lim_{x\to 0} \frac{tanx}{x} = \lim_{x\to 0} \frac{sinx}{x}.\frac{1}{cosx} = \lim_{x\to 0} \frac{sinx}{x}.\frac{1}{cosx} = \lim_{x\to 0} \frac{sinx}{x}.\lim_{x\to 0} \frac{1}{cosx}$

$$= (1).\frac{1}{\cos(0)} = (1).\frac{1}{1} = 1\,.$$

e. $\lim_{x\to 0} \frac{sin2x}{2x^2+x} = \lim_{x\to 0} \frac{sin2x}{2x(x+\frac{1}{2})} = \lim_{x\to 0} \frac{sin2x}{2x}.\lim_{x\to 0} \frac{1}{x+\frac{1}{2}}$

$$= \lim_{2x\to 0} \frac{sin2x}{2x}.\lim_{x\to 0} \frac{1}{x+\frac{1}{2}} = (1).\frac{1}{0+\frac{1}{2}} = 2\,.$$

f. $\lim_{x\to\infty} xsin\frac{1}{x}$

Let $y = \frac{1}{x}$

as $x \to \infty \Longrightarrow y = \frac{1}{x} \longrightarrow 0$

$$\lim_{x\to\infty} x\, sin\frac{1}{x} = \lim_{x\longrightarrow\infty} \frac{sin\frac{1}{x}}{\frac{1}{x}} = \lim_{y\to 0} \frac{siny}{y} = 1$$

g. $\operatorname{Lim}_{x\to 0} \frac{sinx}{|x|}$

$$= \lim_{x\longrightarrow 0^+} \frac{sinx}{|x|} = \lim_{x\longrightarrow 0^+} \frac{sinx}{x} = 1 \text{ and } \lim_{x\longrightarrow 0^-} \frac{sinx}{|x|} = \lim_{x\longrightarrow 0^-} \frac{sinx}{-x} = -1$$

Since $\lim_{x\longrightarrow 0^+} \frac{sinx}{|x|} \neq \lim_{x\longrightarrow 0^-} \frac{sinx}{|x|}$, $\therefore \lim_{x\longrightarrow 0} \frac{sinx}{|x|}$ does not exist

Defn. (Continuous Function)

A function $y = f(x)$ is said to be cont. at x=a if

1. *f(a)* is define
2. $\lim_{x\longrightarrow a}$

Ex.(15)

a. Every Polynomial of the form

$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is a cont. for all x

b. $f(x) = \frac{1}{x}$

c. $f(x) = \frac{x+3}{(x-5)(x+2)}$,f(x) is dis count. at x=5,-2

d. $f(x) = \frac{sinx}{x}$,f(x) is dis count. at x=0

e. $f(x) = \begin{cases} \frac{sinx}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

$f(x)$ is cont. at x=0

$(x) = \begin{cases} \frac{x^2+x-6}{x^2-4}, & x \neq 2 \\ \frac{5}{4}, & x = 2 \end{cases}$

$f(x)$is cont. at x=2

Derivatives المشتـــــقــــات

$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ exist and define, we call it the derivative of f at x (or f is differentiable at x) and denoted by

$\frac{dy}{dx}, \frac{df}{dx}$. That is

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}, \Delta x \neq 0$$

Rules of Derivatives قـــواعـد الاشتقـاق

Rule (1) If y=f(x)=C, where C is a constant, then $\frac{dy}{dx} = f'(x) = 0$

Rule (2) If is +ive integer and y=f(x)= x^n, , then

$$f'(x) = \frac{dy}{dx} = nx^{n-1}$$

Rule (3) If f(x)=c u(x), where c is a constant and u(x) is a differentiable of x then $f'(x) = cu'(x)$ or $\frac{dy}{dx} = c\frac{du}{dx}$

Rule (4) If $u_i(x)$,i=1,2,3,…,n are differentiable functions of x and

$f(x) = u_1(x) \pm u_2(x) \dots \pm u_n(x), then$

$f'(x) = u'_1(x) \pm u'_2(x) \dots \pm u'_n(x).$

Rule (5) If y=f(x)=u(x).v(x), where u(x) and v(x) are differential functions of x, then $f'(x) = u(x)v'(x) + v(x)u'(x)$

Or

$$\frac{dy}{dx} = u(x).\frac{dv}{dx} + v(x).\frac{du}{dv}$$

Rule (6) If $f(x) = \frac{u(x)}{v(x)}, v(x) \neq 0$ Where $u(x)$ and $v(x)$ are differentiable functions of x then

$f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}$ or $\frac{dy}{dx} = \frac{v(x)\frac{du}{dx} - u(x)\frac{dv}{dx}}{[v(x)]^2}$

Rule (7) If f(x)=$[u(x)]^n$ where n is +ive integer and u(x) is a differentiable function of x, then

$f'(x) = n[u(x)]^{n-1}u'(x) or \frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}$

$$f'(x) = \frac{p}{q}[u(x)]^{\frac{p}{q}-1}u'(x)$$

Implicit Differentiations: الاشتــقـــاق الضـمـنـي

Consider the function defined by the eq.f(x,y)=0 which may or may not be solved for y in terms of x.

For example $y - x^3 + 2x - 5 = 0$ can be written as

$y = x^3 - 2x + 5$ and $\frac{dy}{dx} = 3x^2 - 2$

While $y^5 + 4x^2y^2 + x^3 - 2 = 0$ cannot be solved for y in terms of x

Implicit differentiation enables us to find the derivative of such functions whenever they exist

Ex. Find $\frac{dy}{dx}$ **if** $\boldsymbol{y^3 - 3x^2y + x^3 = 5}$

$$3y^2\frac{dy}{dx} - \left[3x^2\frac{dy}{dx} + y(6x)\right] + 3x^2 = 0$$

$$3y^2\frac{dy}{dx} - 3x^2\frac{dy}{dx} - 6xy + 3x^2 = 0$$

$$\left(3y^2 - 3x^2\right)\frac{dy}{dx} = 6xy - 3x^2 \Rightarrow \frac{dy}{dx} = \frac{3\left(2xy - x^2\right)}{3\left(y^2 - x^2\right)} = \frac{2xy - x}{y^2 - x^2}$$

The Second and Higher Derivatives

Given the functions $y = f(x)$. The derivative

$y' = f'(x) = \frac{dy}{dx} = \frac{df}{dx}$ is the 1st derivative of y w.r.t x and

$y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{d^2f}{dx^2}$ is called the 2nd derivative w.r.t x

Thus the second derivative is the derivative of the first derivative. That is

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

In general, if y=f(x) is a differentiable function of x, then the nth derivative of y w.r.t. x is denoted by:

$$f^{(n)}(x) = y^{(n)} = \frac{d^ny}{dx^n}$$

Ex. $If\ y = (3x^3 + 2x - 1)^{1/2}$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

Solu.

$$\frac{dy}{dx} = \frac{1}{2}(3x^3 + 2x - 1)^{-\frac{1}{2}}(9x^2 + 2)$$

$$\frac{d^2y}{dx^2} = \frac{1}{2}(3x^3 + 2x - 1)^{-\frac{1}{2}}(18x) + (9x^2 + 2)\left[-\frac{1}{2}(3x^3 + 2x - 1)^{-\frac{3}{2}}(9x^2 + 2)\right]$$

Ex. If $y = 3x^4 - 5x^3 + 6x - 7$ Find $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots, \frac{d^ny}{dx^n}$

Solu.

$$\frac{dy}{dx} = 12x^3 - 15x^2 + 6$$

$$\frac{d^2y}{dx^2} = 36x^2 - 30x \implies \frac{d^3y}{dx^3} = 72x - 30 \implies \frac{d^4y}{dx^4} = 72$$

$$\frac{d^5y}{dx^5} = \frac{d^6y}{dx^6} = \cdots = \frac{d^ny}{dx^n} = 0$$

H.w

Ex. If $x^2 - y^2 = 1$ Show that $\frac{dy}{dx} = \frac{x}{y}$ and $\frac{d^2y}{dx^2} = -\frac{1}{y^3}$

Ex. If $y = (2x^2 - 5x^{-2})^{-5}$. Find $\frac{dy}{dx}$

$$\frac{dy}{dx} = -5(2x^2 - 5x^{-2})^{-6}(4x + 10x^{-3})$$

H.w

Ex. Find $\frac{d}{dx}\left[\frac{(x^2+x+1)^3}{(x^3+1)^4}\right]$

H.w

Ex. Find $\frac{dy}{dx}, if\ y = \frac{2x^3+3x-1}{x^2+1}$

H.w

Ex. Find $\frac{dy}{dx}, if\ y = (x^2 + 2)(x^3 + 3x + 1)$

Chain Rule and Parametric Equations

Chain Rule If y is a function of x, say y=f(x) and x is a function of t, say x=g(t), then y is a function of t and

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \qquad ..(1)$$

How we obtain eq.(1)

Since

$y = f(x) and\ x = g(t), then$

$y = f[g(t)]$ and $\frac{dy}{dt} = f'[g(t)].g'(t),$

But $x = g(t)$ $and \frac{dx}{dt} = g'(t)$

$$then\ \frac{dy}{dt} = f'(x).\frac{dx}{dt} = \frac{dy}{dx}.\frac{dx}{dt}$$

Ex. If $y = x^3 - 2x^2 + 3$ and $x = t^2 + 2$. Find $\frac{dy}{dt}$ at t=2

Solu.

$$\frac{dy}{dt} = \frac{dy}{dx}.\frac{dx}{dt} = (3x^2 - 4x)(2t)$$

When t=2 $\Longrightarrow x = (2)^2 + 2 = 6$

$$\therefore \left.\frac{dy}{dt}\right|_{t=2} = [3(36) - 4(6)] = (108 - 24)(4) = 336\,.$$

Or

$$\therefore y = (t^2 + 2)^3 - 2(t^2 + 2)^2 + 3 \Longrightarrow \frac{dy}{dt} = 3(t^2 + 2)^2(2t) - 4(t^2 + 2)(2t)$$

$$\therefore \left.\frac{dy}{dt}\right|_{t=2} = 3(36)(4) - 4(6)(4) = 336\,.$$

Parametric Equations

Sometimes, we may describe the curve by expressing both coordinates as functions of a third variable, say $x = g(t)$ and $y = f(t)$

These two eq's are called the parametric eq's. for x and y and the variable t is called a parameter.

Ex. Determine an equation in x and y of the following paramedic's eq.'s and then find $\frac{dy}{dx}$

a. $y = \frac{1}{t}$, x=t $\Rightarrow y = \frac{1}{x} \Rightarrow \frac{dy}{dx} = -\ \frac{1}{x^2}$

b. $y = t^2, x = \frac{t}{1-t}$

$$x = \frac{t}{1-t} \Rightarrow x - xt = t \Rightarrow x = t + xt \Rightarrow x = t(1 + x)$$

$$\Rightarrow t = \frac{x}{1+x} \therefore y = \left(\frac{x}{x+1}\right)^2 \Longrightarrow \frac{dy}{dx} = 2\left(\frac{x}{x+1}\right)\left(\frac{x+1-x}{(x+1)^2}\right)$$

$$\frac{dy}{dx} = \frac{2x}{(x+1)^3}$$

c. $y = 2sint$, x=3cost

$sint = \frac{y}{2}, \;\; cost = \frac{x}{3} \Longrightarrow sin^2t + cos^2t = \frac{y^2}{4} + \frac{x^2}{9} = 1$

$$\Longrightarrow \frac{2y}{4}\frac{dy}{dx} + \frac{2x}{9} = 0 \Longrightarrow \frac{y}{2}\frac{dy}{dx} = -\frac{2}{9}x \Longrightarrow \frac{dy}{dx} = -\frac{4}{9}\frac{x}{y}$$

$(\boldsymbol{d})$ $\;\; y = \frac{5(4-t^2)}{4-t^2} \quad x = \frac{5(4-t^2)}{4-t^2}$

$$y^2 + x^2 = \frac{25(4-t^2)^2}{(4+t^2)^2} + \frac{400t^2}{(4+t^2)^2} = \frac{25(16-8t^2+t^4)+400t^2}{(4+t^2)^2}$$

$$= \frac{25\left(16-8t^2+t^4+16t^2\right)}{(4+t^2)^2} = \frac{25\left(16+8t^2+t^4\right)}{(4+t^2)^2}$$

$$= \frac{25(4+t^2)^2}{(4+t^2)^2}$$

$\therefore y^2 + x^2 = 25 \Longrightarrow 2yy' + 2x = 0 \Longrightarrow y' = \frac{-x}{y}.$

H.w

(e) $y = t^2, x = t - 1$

(f) $y = 2 + 2sint, x = -t + 2cost$

(g) $y = 3tant$, x= $4sect$

(h) $y = sin^3t, x = cos^3t$

(i) $y = t^2 + t - 1, x = t^2 + 2t + 3$

(j) $y = \frac{3(2-t)(2+t)^2}{16t^2+8}, x = \frac{3(2-t)}{6t^2}$

Derivatives of the parametric Eq.'s

The 1st derivative if $y = f(t)$ and $x = g(t)$ $\;,then$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dt}.\frac{dt}{dx} \;\; \Rightarrow \;\; \frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$$

Ex. If $y = t^2 - 1$ and $x = 2t + 3$ Find $\frac{dy}{dt}, \frac{dx}{dt}$ and $\frac{dy}{dx}$

Solu.

$\frac{dy}{dt} = 2t, \quad \frac{dx}{dt} = 2, \frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = (2t).\frac{1}{2} = t$

But

$$x = 2t + 3 \Rightarrow t = \frac{x-3}{2}, \therefore \frac{dy}{dx} = \frac{x-3}{2}$$

Ex.

$y = t - \frac{1}{t}, x = t + \frac{1}{t}$

$$\frac{dy}{dt} = 1 + \frac{1}{t^2}, \frac{dx}{dt} = 1 - \frac{1}{t^2}, \frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}}$$

$$\therefore \frac{dy}{dx} = \frac{t^2 + 1}{t^2 - 1} \Longrightarrow \frac{dy}{dx} = \frac{t + \frac{1}{t}}{t - \frac{1}{t}} = \frac{x}{y}$$

Or

$$y^2 - x^2 = \left(t - \frac{1}{t}\right)^2 - \left(t + \frac{1}{t}\right)^2 = t^2 - 2 + \frac{1}{t^2} - t^2 - 2 - \frac{1}{t^2}$$

$$\therefore y^2 - x^2 = -4 \Longrightarrow 2y\frac{dy}{dx} - 2x = 0 \Longrightarrow 2y\frac{dy}{dx} = 2x$$

$$\therefore \frac{dy}{dx} = \frac{x}{y}$$

The 2nd derivative:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{\frac{dx}{dt}}.\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$$

$$= \frac{1}{\frac{dx}{dt}} \frac{\left(\frac{dx}{dt}\right).\left(\frac{d^2y}{dt^2}\right) - \left(\frac{dy}{dt}\right).\left(\frac{d^2x}{dt^2}\right)}{\left(\frac{dx}{dt}\right)^2}$$

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{dx}{dt}\right)\cdot\left(\frac{d^2y}{dt^2}\right) - \left(\frac{dy}{dt}\right)\cdot\left(\frac{d^2x}{dt^2}\right)}{\left(\frac{dx}{dt}\right)^3}$$

Ex. If $x = t - t^2, y = t - t^3$.Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when t=2

Solu.

$$x = t - t^2 \Longrightarrow \frac{dx}{dt} = 1 - 2t \Longrightarrow \frac{d^2x}{dt^2} = -2$$

$$, y = t - t^3 \Longrightarrow \frac{dy}{dt} = 1 - 3t^2 \Longrightarrow \frac{d^2y}{dt^2} = -6t$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{1-3t^2}{1-2t}\Rightarrow \left.\frac{dy}{dx}\right|_{t=2}=\frac{1-12}{1-4}=\frac{11}{3}$$

$$\frac{d^2y}{dx^2}=\frac{\left(\frac{dx}{dt}\right)\cdot\left(\frac{d^2y}{dt^2}\right)-\left(\frac{dy}{dt}\right)\cdot\left(\frac{d^2x}{dt^2}\right)}{\left(\frac{dx}{dt}\right)^3}$$

$$\frac{d^2y}{dx^2}=\frac{(1-2t)(-6t)-(1-3t^2)(-2)}{(1-2t)^3}=\frac{-6t+12t^2+2-6t^2}{(1-2t)^3}$$

$$=\frac{6t^2-6t+2}{(1-2t)^3}\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=2}=\frac{6(4)-6(2)+2}{(1-4)^3}=\frac{24-12+2}{(-3)^3}$$

$$=\frac{14}{-27}$$

Or

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{1-3t^2}{1-2t}\right)=\frac{\frac{d}{dt}}{\frac{dx}{dt}}\left(\frac{1-3t^2}{1-2t}\right)$$

$$=\frac{\frac{(1-2t)(-6t)-(1-3t^2)(-2)}{(1-2t)^2}}{(1-2t)}$$

$$=\frac{6t^2-6t+2}{(1-2t)^3}\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=-2}=\frac{14}{-27}$$

<u>Note</u> The chain rule can be extended to many variables In general: If $y=f_1(t_1)$, $t_1=f_2(t_2)$, $t_2=f_3(t_3)$,..,

$t_{n-1}=f_n(t_n)$,$t_n=f(x)$, then

$$\frac{dy}{dx}=\frac{dy}{dt_1}\frac{dt_1}{dt_2}.\frac{dt_2}{dt_3}\ldots\ldots\frac{dt_{n-1}}{dt_n}.\frac{dt_n}{dx}$$

Ex. If $y=x^3+2x^2+3x-4$ .Find $\frac{dy^2}{dx^2}$

<u>Solu.</u>

Let $u=y^2$ and $v=x^2$

$\frac{dy^2}{dx^2}=\frac{du}{dv}=\frac{du}{dy}.\frac{dy}{dx}.\frac{dx}{dv}=(2y)(3x^2+4x+3).\frac{1}{2x}$

Ex. If $y=\frac{x}{x^2+1}$. Find $\frac{d\sqrt{y}}{d\sqrt{x}}$

Solu.

Let$u=\sqrt{y}$ and $v=\sqrt{x}$

$$\frac{d\sqrt{y}}{d\sqrt{x}}=\frac{du}{dv}=\frac{du}{dy}.\frac{dy}{dx}.\frac{dx}{dv}=\frac{1}{2\sqrt{y}}.\left(\frac{(x^2+1)(1)-x(2x)}{(x^2+1)^2}\right).2\sqrt{x}$$

Ex. Find $\frac{d\sqrt{x^2+1}}{dx^3}$

Solu.

Let $u=\sqrt{x^2+1}$ and $v=x^3$

$$\frac{du}{dv}=\frac{du}{dx}.\frac{dx}{dv}=\frac{2x}{2\sqrt{x^2+1}}.\frac{1}{3x^2}$$

Indeterminate Forms:

The meaning Less $\frac{0}{0},\frac{\infty}{\infty},0^{\circ},1^{\infty},\infty^0,0.\infty,\infty-\infty$ are Known as indeterminate forms

Sometimes the limit $\lim_{x\to a}\frac{f(x)}{g(x)}$ produce $\frac{0}{0}$ or $\frac{\infty}{\infty}$ when substituting x=a

For example:

$\lim_{x\to 2}\frac{x^2-4}{x-2}=\frac{4-4}{2-2}=\frac{0}{0}$ Meaningless so the solution is

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)}=\lim_{x\to 2}(x+2)=2+2=4\,.$$

L'Hopital's Rule (First Form) قـــاعـــدة اوبــتــال

Suppose f(a)= g(a)=0 or ∞ and $f'(a)$, and $g'(a)$ exist with $g'(a)\neq 0$. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(a)}{g'(a)}=\frac{f'(a)}{g'(a)}\,.$$

Ex.

a) $\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{2x}{1}=2(2)=4$.

b) $\lim_{x\to 0}\frac{x-2x^2}{3x^2+5x}=\lim_{x\to 0}\frac{1-4x}{6x+5}=\frac{1-0}{0+5}=\frac{1}{5}$.

c) $\lim_{x\to\infty}\frac{6x+5}{3x-8}=\lim_{x\to\infty}\frac{6}{3}=\frac{6}{3}=2$.

d) $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}-0}{1}=\lim_{x\to 0}\frac{1}{2\sqrt{1+x}}=\frac{1}{2}$

L'Hopital's Rule (Stronger Form)

Suppose f(a)=g(a)=0 or ∞ and the functions f and g with their derivatives are continuous in some interval I.

To find $\lim_{x\to a}\frac{f(x)}{g(x)}$,we proceed to differentiate f and g as long as

We still get the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. but we stop the differentiation as soon as one or other derivatives is different form 0 or ∞ at x=a

Note

L' Hopital's rule does not apply when either numerator or denominator has a finite non- zero limit

Ex.

a) $\lim_{x\to 1}\frac{x^3-3x+2}{x^3-x^2-x+1} = \lim_{x\to 2}\frac{3x^3-3}{3x^2-2x-1}$

$$\lim_{x\to 1}\frac{6x}{6x-2} = \frac{6(1)}{6(1)-2} = \frac{6}{4} = \frac{3}{2}.$$

b) $\lim_{x\to 0}\frac{\sqrt{1+x}-\frac{1}{2}x-1}{x^2} = \lim_{x\to 0}\frac{\frac{1}{2}(1+x)^{-1/2}-\frac{1}{2}}{2x}$

$$= \lim_{x\to 0}\frac{-\frac{1}{4}(1+x)^{-3/2}}{2} = -\frac{1}{8}\ .$$

c) $\lim_{x\to\infty}\frac{2x^3-x^2+3x+1}{x^3+2x^2-x-1} = \lim_{x\to\infty}\frac{6x^2-2x+3}{3x^2+4x-1}$

$$\lim_{x\to\infty}\frac{12x-2}{6x+4} = \lim_{x\to\infty}\frac{12}{6} = \frac{12}{6} = 2\ .$$

"Transcendental Functions"

Trigonometric Functions

Theorem (1): $\boldsymbol{if\ y = f(x) = \sin x\ \ then\ \frac{dy}{dx} = \acute{f}(x) = \cos x}$

Proof

Let $h = \Delta x$

$$\acute{f}(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0}\frac{}{h}$$

$$= \lim_{h\to 0}[\frac{-\sin x\,(1-\cos h)}{h} + \frac{\cos x \sin h}{h}]$$

$$= -\sin x \lim_{h\to 0}\frac{1-\cos h}{h} + \cos x \lim_{h\to 0}\frac{\sin h}{h}$$

$$= -\sin x\,(0) + \cos x\,.\,(1) = \cos x\,.$$

Theorem (2): $\boldsymbol{if\ y = f(x) = \cos x\ \ , then\ \frac{dy}{dx} = \acute{f} = -\sin x}$

Proof

Let $h = \Delta x$

$$\acute{f}(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0}\frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h\to 0}[\frac{-\cos x\,(1-\cos h)}{h} - \frac{\sin x \sin h}{}]$$

$$= \lim_{h\to 0}[\frac{-\cos x\,(1-\cos h)}{h} - \lim_{h\to 0}\frac{}{h}]$$

$$= -\cos x \lim_{h\to 0}\frac{1-\cos h}{h} - \sin x \lim_{h\to 0}\frac{\sin h}{h} = 1$$

$$= -\cos x\,(0) - \sin x\,(1) = -\sin x\ \ .$$

<u>Theorem (3)</u>

(1) If $y = \tan x \quad then\ \frac{dy}{dx} = \sec^2 x$

(2) If $y = \cot x \quad then\ \frac{dy}{dx} = -\ {}^{2} x$

(3) If $y = \sec x \quad then\ \frac{dy}{dx} = \sec x \tan x$

(4) If $y = \csc x \quad then\ \frac{dy}{dx} = -\csc x \cot x$

Proof (1)

$$y = \tan x = \frac{\sin x}{\cos x} \Longrightarrow \frac{dy}{dx} = \frac{\cos x \,.\cos x - \sin x\,(-\sin x)}{\cos^2 x}$$

$$\frac{dy}{dx} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \ .$$

Now, if u=u(x) is a differentiable function of x and

1. $y = \sin u \quad then\ \frac{dy}{dx} = \cos u \,.\frac{du}{dx}$
2. $y = \cos u \quad then\ \frac{dy}{dx} = -\sin u \,.\frac{du}{dx}$
3. $y = \tan u \quad then\ \frac{dy}{dx} = \sec^2 u \,.\frac{du}{dx}$
4. $y = \cot u \quad then\ \frac{dy}{dx} = -\csc^2 u \,.\frac{du}{dx}$
5. $y = \sec u \quad then\ \frac{dy}{dx} = \sec u \tan u \,.\frac{du}{dx}$
6. $y = \csc u \quad then\ \frac{dy}{dx} = -\csc u \cot u \,.\frac{du}{dx}$

Ex (1) find $\frac{dy}{dx}$ of the following

(1) $y = \sin(x^2 + 2x - 5) \Longrightarrow \frac{dy}{dx} = \cos(x^2 + 2x - 5).(2x + 2)$

$\quad = 2(x + 1)\cos(x^2 + 2x - 5)$

(2) $y = \sin^2(x^2 + \frac{1}{x^2}) \Longrightarrow \frac{dy}{dx} = 2\sin(x^2 + \frac{1}{x^2}) \,.\cos(x^2 + \frac{1}{x^2})(2x - \frac{2}{x})$

(3) $y = \tan(2x) \,.\cos(x^2 + 1)$

$$\frac{dy}{dx} = -\tan(2x)\sin(x^2 + 1)(2x) + \cos(x^2 + 1)\sec^2(2x) \,.2$$

(4) $y = \tan^{-3}(3x^2 + \sec^2 2x)$

$$\frac{dy}{dx} = -3\tan^{-4}(3x^2 + \sec^2 2x) \,.\sec^2(3x^2 + \sec^2 2x).(6x + 2\sec 2x\ \sec 2x \tan 2x \,.2)$$

(5) $y = \frac{\sec[\sin(2x+1)]}{\tan(x^3+1)}$

$$\frac{dy}{dx} = \frac{\tan(x^3+1)\sec[\sin(2x+1)].\tan[\sin(2x+1)].\cos(2x+1)2}{\tan^2(x^3+1)}$$

$$\frac{-\sec[\sin(2x+1)].\sec^2(x^3+1).3x^2}{\tan^2(x^3+1)}$$

Ex. (2) Find $\frac{d}{dx}[\sec^{-2}(x^2+2x) - \tan^2(\ 3x)]$

$= -2\sec^{-3}(x^2+2x)\sec(x^2+2x)\tan(x^2+2x)(2x+2) - 2\tan(\sin 3x)$

$\sec^2(\sin 3x).\cos(3x).3$

Ex (3) Find $\frac{dy}{dx}$ *if* $x^2 + 5x - \tan^2(xy) = 10$

$$2x + 5 - 2\tan(xy)\sec^2(xy)\left(x\frac{dy}{dx} + y.(1)\right) = 0$$

$$\frac{dy}{dx} = \frac{2x + 5 - 2y\tan(xy)\sec^2(xy)}{2x\tan(xy)\sec^2(xy)}.$$

Ex. (4) Find the equation of tangent to the curve $x\sin 2y = y\cos 2x$ at point $(\frac{\pi}{4}, \frac{\pi}{2})$

<u>Soul.</u>

$$2x\cos 2y.\frac{dy}{dx} + \sin 2y = -2y\sin 2x + \cos 2x.\frac{dy}{dx}$$

$$2x\cos 2y.\frac{dy}{dx} - \cos 2x\frac{dy}{dx} = -2y\sin 2x - \sin 2y$$

$$(2x\ \cos 2y - \cos 2x)\frac{dy}{dx} = -2y\sin 2x - \sin 2y$$

$$\frac{dy}{dx} = \frac{2y\sin 2x + \sin 2y}{\cos 2x - 2x\cos 2y} \quad \textit{of tangent at any } p\ (x, y)$$

$$m = \frac{dy}{dx}\ \ at\ x = \frac{\pi}{4}\ \ , y = \frac{\pi}{2}$$

Is $\quad m = \frac{2\left(\frac{\pi}{2}\right)\sin\frac{\pi}{2} + \sin\pi}{\cos\frac{\pi}{2} - 2\left(\frac{\pi}{4}\right)\cos\pi} = \frac{\pi(1)+0}{0-\frac{\pi}{2}(-1)}$

$$m = 2 \implies m = \frac{y - y1}{x - x1} \implies 2 = \frac{y - \frac{\pi}{2}}{x - \frac{\pi}{4}} \quad \textit{is the required eq.}$$

Ex. (5) If $\acute{f} = \sin x^2$ *and* $y = f\left(\frac{2x+1}{x+1}\right)$ *find* $\frac{dy}{dx}$.

<u>Soul.</u>

$$y = f\left(\frac{2x+1}{x+1}\right) \implies \frac{dy}{dx} = \acute{f}\left(\frac{2x+1}{x+1}\right) . \frac{(x+1)(2) - (2x+1)(1)}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{1}{(x+1)^2}\acute{f}\left(\frac{2x+1}{x+1}\right) = \frac{1}{(x+1)^2}\sin(\frac{2x+1}{x+1})^2 .$$

Ex. (6) If $y = \tan^{-3}(\sin 2x)$ *find* $\frac{dy^2}{dx^2}$ chain rule

$$\frac{dy^2}{dx^2} = \frac{du}{dv} = \frac{du}{dy} . \frac{dy}{dx} . \frac{dx}{dv}$$

$$= 2y[-3\tan^{-4}(\sin 2x)\sec^2(\sin 2x)\cos(2x). 2]. \frac{1}{2x} .$$

Ex. (7) If $y = \sec 2t$ *and* $x = \csc 2t$ *find* $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *at* $t = \frac{\pi}{6}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\sec 2t \tan 2t}{-2\csc 2t \cot 2t} = -\tan^3 2t$$

$$\left.\frac{dy}{dx}\right| t = \frac{\pi}{6} = -\tan^3(\frac{\pi}{3}) = -(\sqrt{3})^3 = -3\sqrt{3}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}}{\frac{dx}{dt}}(-\tan^3 2t) = \frac{-3\tan^2 2t . \sec^2 2t . 2}{-2\csc 2t \cot 2t}$$

$$= -3\tan^4 2t \sec 2t$$

$$\left.\frac{d^2y}{dx^2}\right| = -3\tan^4\left(\frac{\pi}{3}\right)\sec\left(\frac{\pi}{3}\right) = -3\left(\sqrt{3}\right)^4 . 2 = -54 \ .$$

$$t = \frac{\pi}{6}$$

Ex. (8) evaluate the following limits:

1. $\lim\limits_{x\to 0}\frac{\sin 3x}{\sin 5x}=\lim\limits_{x\to 0}\frac{3\cos 3x}{5\cos 5x}=\frac{3\cos 0}{5\cos 0}=\frac{3}{5}.$

2. $\lim\limits_{x\to 0}\frac{\sin 2x}{2x^2+x}=\lim\limits_{x\to 0}\frac{2\cos 2x}{4x+1}=\frac{2\cos 0}{0+1}=2\ .$

3. $\lim\limits_{x\to 0}\frac{\tan 3x}{\sin x}=\lim\limits_{x\to 0}\frac{3\sec^2 3x}{\cos x}=\lim\limits_{x\to 0}\frac{3\sec^2 0}{\cos 0}=3\ .$

4. $\lim\limits_{x\to 0}\frac{x-\sin x}{x^3}=\lim\limits_{x\to 0}\frac{1-\cos x}{3x^2}=\lim\limits_{x\to 0}\frac{\sin x}{6x}$

$=\lim\limits_{x\to 0}\frac{\cos x}{6}=\frac{\cos 0}{6}=\frac{1}{6}\ .$

5. $\lim\limits_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\infty-\infty\ meaningless$

$=\lim\limits_{x\to 0}\frac{x-\sin x}{x\sin x}=\lim\limits_{x\to 0}\frac{1-\cos x}{x\cos x+\sin x\,(1)}$

$=\frac{1-\cos(0)}{0+\sin(0)}=\frac{1-1}{0+0}=\frac{0}{0}\ meaningless$

$\lim\limits_{x\to 0}\frac{\sin x}{\qquad}=\frac{\sin(0)}{0+\cos(0)+\cos(0)}=\frac{0}{0+1+1}=\frac{0}{2}$

$=0\,.$

6. $\lim\limits_{x\to 0}\frac{\sin x^2}{x\sin x}=\frac{0}{0}\ meaningless$

$=\lim\limits_{x\to 0}\frac{\cos x^2\,(2x)}{x\cos x+\sin x\,.(1)}=\frac{0}{0}\ meaning\ less$

$\lim\limits_{x\to 0}\frac{\cos x^2\,(2)+2x(-\sin x^2\,(2x))}{-x\sin x+\cos x+\cos x}=\frac{2+0}{0+1+1}=\frac{2}{2}=1\ .$

7. $\lim\limits_{x\to\frac{\pi}{2}}(\sec x-\tan x)=\infty-\infty$

$=\lim\limits_{x\to\frac{\pi}{2}}\left(\frac{1}{\cos x}-\frac{\sin x}{\cos x}\right)=\lim\limits_{x\to\frac{\pi}{2}}\frac{1-\sin x}{\cos x}$

$=\lim\limits_{x\to\frac{\pi}{2}}\frac{-\cos x}{-\sin x}=\lim\limits_{x\to\frac{\pi}{2}}\frac{\cos x}{\sin x}=\frac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}}=\frac{0}{1}=0\ .$

The Inverse of Trigonometric Functions

Defn.

(1) For -1≤x≤1, we define the no. $y=f(x)=\sin^{-1}x\ \ for\ which$

$-\frac{\pi}{2}\le y\le\frac{\pi}{2}\quad and\ x=\sin y\ .$

(2) For -1≤x≤1, we define the no. $y=f(x)=\cos^{-1}x\ \ for\ which$

$0\le y\le\pi\quad and\ x=\cos y\ \ .$

(3) For $-\infty<x<\infty$, we define the no. $y=f(x)=\tan^{-1}x\ \ for\ which$

$-\frac{\pi}{2}<y<\frac{\pi}{2}\ \ and\ \ x=\tan y$

(4) For $-\infty < x < \infty$, we define the no. $y = f(x) = \cot^{-1} x \ \ for\ which$

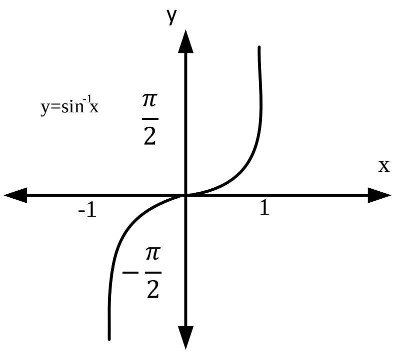
$$0 < y < \pi \quad and\ x = \cot y\ .$$

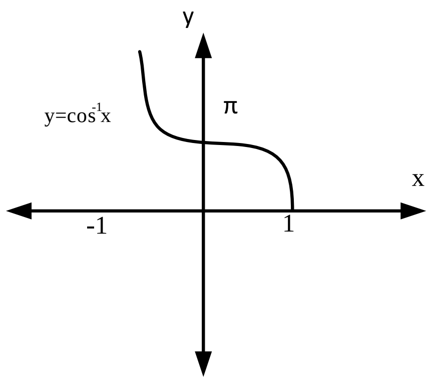
(5) For $x \leq -1\ \ or\ x \geq 1$, we define the no. $y = f(x) = \sec^{-1} x\ for\ which$

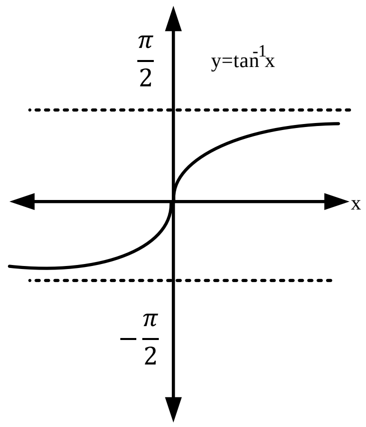
$$-\pi \leq y < -\frac{\pi}{2}\ or\ o \leq y < \frac{\pi}{2}\ and\ x = \sec y$$

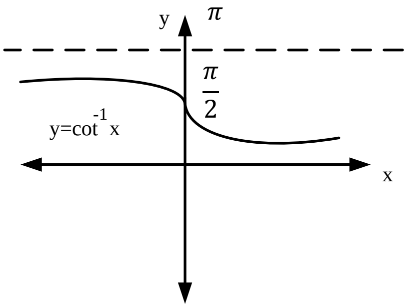
(6) For $x \leq -1\ or\ x \geq 1$, we define the no. $y = f(x) = \csc^{-1} x\ \ for\ which$

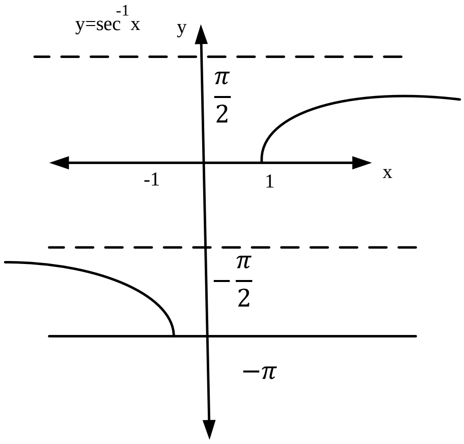
$$-\pi < y \leq -\frac{\pi}{2} \qquad\qquad {}_2\ \ and\ x = \csc y\ .$$

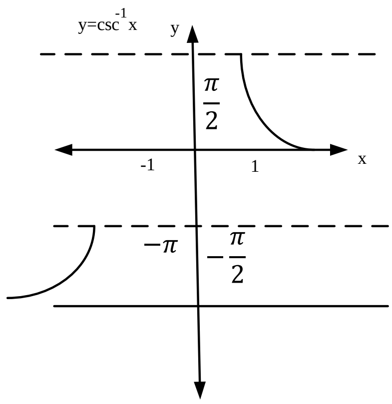












Note: $\boldsymbol{\sin^{-1} x \neq \frac{1}{\sin x}}$, $\boldsymbol{(\sin x)^{-1} = \frac{1}{\sin x} = \csc x}$.

Some Important Properties of The Inverse of Trigonometric Functions:

(1) $\sin^{-1}(-x) = -\sin^{-1} x$

(2) $\cos^{-1}(-x) = \pi - \cos^{-1} x$

(3) $\tan^{-1}(-x) = -\tan^{-1} x$

(4) $\cot^{-1}(-x) = -\cot^{-1} x$

(5) $\sec^{-1}(-x) = \pi - \sec^{-1} x$

(6) $\csc^{-1}(-x) = -\csc^{-1} x$

(7) $\sin^{-1} x = \frac{\pi}{2} - \cos^{-1} x$
(8) $\tan^{-1} x = \frac{\pi}{2} - \cot^{-1} x$
(9) $\sec^{-1} x = \frac{\pi}{2} - \csc^{-1} x$
(10) $\sin^{-1} x = \csc^{-1} \frac{1}{x}$
(11) $\cos^{-1} x = \sec^{-1} \frac{1}{x}$
(12) $\tan^{-1} x = \cot^{-1} \frac{1}{x}$
(13) $\sin^{-1}(\sin x) = x$
(14)

Proof (1)

Proof (2)

Let $y = \cos^{-1}(-x) \Longrightarrow -x = \cos y \Longrightarrow x = -\cos y \Longrightarrow$

$x = \cos(\pi - y) \Longrightarrow \pi - y = \cos^{-1} x \Longrightarrow y = \pi - \cos^{-1} x$.

Proof (3)

Let $y = \tan^{-1}(-x) \Longrightarrow -x = \tan y \Longrightarrow x = -\tan y$

$\Longrightarrow x = \tan(-y) \Longrightarrow -y = \tan^{-1} x \Longrightarrow y = -\tan^{-1} x$.

Proof (7)

Let $y = \sin^{-1} x \Longrightarrow x = \sin y \Longrightarrow x = \cos(\frac{\pi}{2} - y) \Longrightarrow \frac{\pi}{2} - y = \cos^{-1} x$

$\Longrightarrow y = \frac{\pi}{2} - \cos^{-1} x$.

Proof (10)

Let $y = \sin^{-1} x \Longrightarrow x = \sin y \Longrightarrow x = \frac{1}{\csc y} \Longrightarrow \frac{1}{x} = \csc y$

$\Longrightarrow y = \cos^{-1}\left(\frac{1}{x}\right)$.

Proof (13)

Let $y = \sin^{-1}(\sin x) \Rightarrow \sin y = \sin x \Rightarrow x = y.$

Ex. (1) If $\boldsymbol{\theta = \sin^{-1}\frac{\sqrt{3}}{2}}$ $\boldsymbol{find\ \cos\theta, \tan\theta, \cot\theta, \sec\theta\ \ and\ \csc\theta}$

<u>Soul .</u>

$$\theta = \sin^{-1}\frac{\sqrt{3}}{2} \Rightarrow \frac{\sqrt{3}}{2} = \sin\theta \Rightarrow \theta = \frac{\pi}{3}$$

$$\cos\frac{\pi}{3} = \cos\theta\frac{1}{2}\ \ and\ so\ on\ ....\quad .$$

Ex. (2) Evaluate

$(\mathbf{1})\ \mathbf{\sin(\cos^{-1}\frac{1}{\sqrt{2}})}$ $\qquad (\mathbf{2})\ \mathbf{\sec(\cos^{-1}\frac{1}{2})}$

$(\mathbf{3})\ \mathbf{\tan(\sin^{-1}(-\frac{1}{2}))}$ $\qquad (\mathbf{4})\ \mathbf{\sin(\sin^{-1}\frac{\pi}{3})}$

1. Let $\theta = \cos^{-1}\frac{1}{\sqrt{2}} \Rightarrow \frac{1}{\sqrt{2}} = \cos\theta \Rightarrow \theta = \frac{\pi}{4}$

So $\quad \sin(\cos^{-1}\frac{1}{\sqrt{2}}) = \sin\theta = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

2. Let $\theta = \cos^{-1}\frac{1}{2} \Rightarrow \frac{1}{2} = \cos\theta \Rightarrow \theta = \frac{\pi}{3}$

So $\quad \sec(\cos^{-1}\frac{1}{2}) = \sec(\theta) = \sec(\frac{\pi}{3}) = 2$.

3. Let $\theta = \sin^{-1}(-\frac{1}{2}) \Rightarrow \sin\theta = \frac{-1}{2} \Rightarrow \theta = -\frac{\pi}{6}$

So $\quad \tan(\sin^{-1}(-\frac{1}{2})) = \tan(\theta) = \tan(-\frac{\pi}{6}) = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$.

4. $\sin(\sin^{-1}\frac{\pi}{3}) = \frac{\pi}{3}$.

Ex. (3) Solve for x if $\boldsymbol{\tan^{-1}x - \cot^{-1}x = \frac{\pi}{4}}$

<u>Soul .</u>

$$\tan^{-1}x - \left(\frac{\pi}{2} - \tan^{-1}x\right) = \frac{\pi}{4} \Rightarrow \tan^{-1}x - \frac{\pi}{2} + \tan^{-1}x = \frac{\pi}{4}$$

$$\Rightarrow 2\tan^{-1}x = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$$

$$\Rightarrow \tan^{-1}x = \frac{3\pi}{8} \Rightarrow x = \tan\frac{3\pi}{8}$$

$$x = \tan\frac{3\pi}{8} = \tan(\frac{\frac{3\pi}{4}}{2}) = \frac{\sin\frac{3\pi}{4}}{1+\cos\frac{3\pi}{4}} = \frac{\sin(\frac{\pi}{2}+\frac{\pi}{4})}{1+\cos(\frac{\pi}{2}+\frac{\pi}{4})}$$

$$= \frac{\cos\frac{\pi}{4}}{1-\sin\frac{\pi}{4}} = \frac{\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}-1}$$

Note:- $\tan\frac{x}{2} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$

Derivative of The Inverse of Trigonometric Functions

مشتقة معكوس الدوال المثلثية

Theorem:

(1) If $y = \sin^{-1} x$ *then* $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$.

(2) If $y = \cos^{-1} x$ *then* $\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$.

(3) If $y = \tan^{-1} x$ *then* $\frac{dy}{dx} = \frac{1}{1+x^2}$.

(4) If $y = \cot^{-1} x$ *then* $\frac{dy}{dx} = \frac{-1}{1+x^2}$.

(5) If $y = \sec^{-1} x$ *then* $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$.

(6) If $y = \csc^{-1} x$ *then* $\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}}$.

Proof (1) $y = \sin^{-1} x \Rightarrow x = \sin y \Rightarrow 1 = \cos y \frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}} .$$

Proof (2) $y = \cos^{-1} x \Rightarrow x = \cos y \Rightarrow 1 = -\sin y \frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}} .$$

Proof (3) $y = \tan^{-1} x \Rightarrow x = \tan y \Rightarrow 1 = \sec^2 y \frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2} .$$

Now, if u=u(x) is a differentiable function of x and

1. $y = \sin^{-1} u \ then \ \frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dy}$
2. $y = \cos^{-1} u \ then \ \frac{dy}{dx} = \frac{-1}{\sqrt{1-u^1}}\frac{du}{dx}$
3. $y = \tan^{-1} u \ then \ \frac{dy}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$
4. $y = \cot^{-1} u \ then \ \frac{dy}{dx} = \frac{-1}{1+u^2}\frac{du}{dx}$
5. $y = \sec^{-1} u \ then \ \frac{dy}{dx} = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$
6. $y = \csc^{-1} u \ then \ \frac{dy}{dx} = \frac{-1}{u\sqrt{u^2-1}}\frac{du}{dx}$

Ex. (4) Find $\frac{dy}{dx}$ of the following functions:

(1) $y = \sin^{-1}(x^2 + 3x - 1) \Rightarrow \frac{dy}{dx} = \frac{(2x+3)}{\sqrt{1-(x^2+3x-1)^2}}$

(2) $y = x^2 \tan^{-1}\sqrt{x} \Rightarrow \frac{dy}{dx} = x^2 \frac{1}{1+x}.\frac{1}{2\sqrt{x}} + 2x\tan^{-1}\sqrt{x}$

(3) $y = \cos^{-1}(x^2 + \tan^{-1} 3x) \Rightarrow \frac{dy}{dx} = -\frac{1}{\sqrt{1-(x^2+\tan^{-1}3x)}}.(2x + \frac{3}{1+9x^2})$

(4) $y = \sin^2(\sec^{-1} 2x).\cot^{-1}(\frac{1}{x})$

$$\frac{dy}{dx} = \sin^2(\sec^{-1} 2x).\frac{-1}{1+(\frac{1}{x})^2}.\left(-\frac{1}{x^2}\right) + \cot^{-1}(\frac{1}{x}).2\sin(\sec^{-1} 2x)$$

$$\cos(\sec^{-1} 2x).\frac{2}{2x\sqrt{4x^2-1}}$$

Ex. (5) Find $\frac{d}{dx}[\frac{\tan^2(3x^2+1)}{\sin^{-1}(x^2-1)}]$

$$= \frac{\sin^{-1}(x^2-1).2\tan(3x^2+1)\sec^2(3x^2+1)(6x) - \tan^2(3x^2+1)\frac{(2x)}{\sqrt{1-(x^2-1)^2}}}{[\sin^{-1}(x^2-1)]^2}$$

Ex. (6) If $y = \sin^{-1}(\frac{x-1}{x+1})$ $find$ $\frac{dy^3}{d\sec 2x}$

Let $u = y^3 \Rightarrow \frac{du}{dy} = 3y^2$

$v = \sec 2x \Rightarrow \frac{dv}{dx} = 2\sec 2x \tan 2x$

$$\frac{dy^3}{d\sec 2x}=\frac{du}{dv}=\frac{du}{dy}\cdot\frac{dy}{dx}\cdot\frac{dx}{dv}=3y^2\cdot\frac{\frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}}{\sqrt{1-(\frac{1-x}{1+x})^2}}\cdot\frac{1}{2\sec 2x\tan 2x}\,.$$

Ex. (7) If $y=\sin^{-1}t$ and $x=\cos^{-1}t\,.Find$ $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $t=\frac{1}{2}$

$$y=\sin^{-1}t\Rightarrow\frac{dy}{dt}=\frac{1}{\sqrt{1-t^2}},x=\cos^{-1}t\Rightarrow\frac{dx}{dt}=\frac{-1}{\sqrt{1-t^2}}$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=-1\;and\;\;\frac{d^2y}{dx^2}=0$$

H.W

Ex. (8) Show that the functions

$$f(x)=\sin^{-1}(\frac{x-1}{x+1})\;and\;g(x)=2\tan^{-1}\sqrt{x}$$

Have the same derivative.

Ex. (9): Evaluate the following limits:

(1) $\lim_{x\longrightarrow 0}\frac{\sin^{-1}x}{x}=\lim_{x\longrightarrow 0}\frac{1}{\sqrt{1-x^2}}=1$.

(2) $\lim_{x\longrightarrow 0}\frac{2\tan^{-1}3x}{5x}=\lim_{x\longrightarrow 0}\frac{2\frac{3}{1+(3x)^2}}{5}=\frac{6}{5}$.

(3) $\lim_{x\to 0}\frac{\tan^{-1}x-x}{x^3}=\lim_{x\to 0}\frac{\frac{1}{1+x^2}-1}{3x^2}=\lim_{x\to 0}\frac{\frac{(1+x^2)(0)-1(2x)}{(1+x^2)^2}}{6x}$

$$=\lim_{x\to\to 0}\frac{\frac{-2x}{(1+x^2)^2}}{6x}=\lim_{x\to 0}\frac{\frac{(1+x^2)^2(-2)-(-2x)2(1+x^2)(2x)}{(1+x^2)^4}}{6}$$

$$=\frac{\frac{-2-0}{1}}{6}=\frac{-2}{6}=-\frac{1}{3}\,.$$

H.W

(4) $\lim_{x\to 0}\frac{\sin^{-1}x-x}{x^3}=\frac{1}{6}$ استعمال طريقة اوبتال ثلاث مرات

The Logarithmic Function

الدالة اللوغارتمية

The logarithmic was discovered by a Scottish Nobleman John Napier (1550-1617)

$y = f(x) = \log_b x \Leftrightarrow x = b^y$ *where* y *is the logarithm*

x *is the number*

b *is the base*

If b=10, we write $y = \log_{10} x$ *or* $y = \log x$

If b=e =2.7183, we write $y = \log_e x$ *or* $y = \ln x$

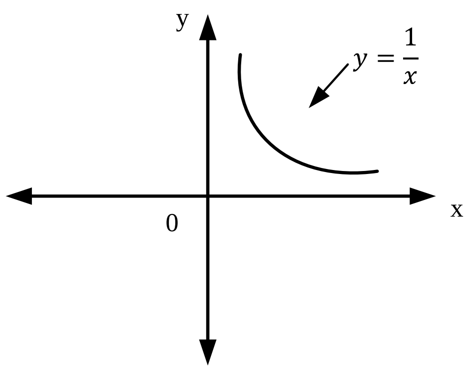
In read natural logarithm (اللوغاريتم الطبيعي)

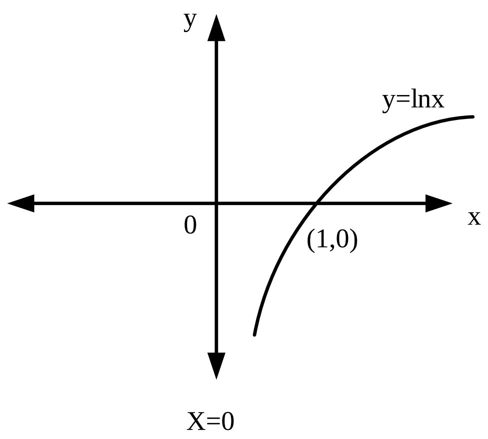
Relation Between The Logarithm and The Natural Logarithm

Let $y = \log_b x \Leftrightarrow x = b^y \Longrightarrow \ln x = \ln b^y = y \ln b \Longrightarrow y = \frac{\ln x}{\ln b}$

So $\log_b x = \frac{\ln x}{\ln b}$

<u>Defn.</u> For x>0, we define $\ln x = \int_1^x \frac{dt}{t}$





Properties:

1. $\ln(a.b) = \ln a + \ln b$
2. $\ln(\frac{a}{b}) = \ln(a) - \ln(b)$
3. $\ln 1 = 0$
4. $\ln a^r = r \ln a$ *where* $r = \frac{p}{q}$ q *and* $\neq 0$ *to are integers*

Proof (1)

$$\ln(a.b) = \int_1^{a.b} \frac{dt}{t} = \int_1^{a} \frac{dt}{t} + \int_a^{a.b} \frac{dt}{t} = \ln a + \int_a^{a.b} \frac{dt}{t}$$

Let $u = \frac{t}{a} \Longrightarrow au = t \Longrightarrow a\,du = dt$

$$\int_a^{a.b} \frac{dt}{t} = \int_1^{b} \frac{a\,du}{at} = \int_1^{b} \frac{du}{u} = \ln(b)\,.$$

$\ln(a.b) = \ln a + \ln b$

Proof (2)

$$a = \frac{a}{b}.b \Longrightarrow \ln(a) = \ln(\frac{a}{b}.b) = \ln\left(\frac{a}{b}\right) + \ln(b)$$

$$\Longrightarrow \ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)\ .$$

Proof (3)

$$1 = \frac{a}{a} \Longrightarrow \ln 1 = \ln\left(\frac{a}{a}\right) = \ln a - \ln a = 0\,.$$

Proof (4)

Let $u = a^{\frac{1}{q}}$ $then\ a^r = a^{\frac{p}{q}} = (a^{\frac{1}{q}})^p = u^p$

Also $a^r = u^p \Longrightarrow \ln a^r = \ln u^p = p \ln u \ ---\ (1)$

$u = a^{\frac{1}{q}} \Longrightarrow a = u^q \Longrightarrow \ln a = q \ln u \Longrightarrow \frac{1}{q}\ln a = \ln u$ ---- (2)

From (1) and (2)

$$\ln a^r = p.\frac{1}{q}\ln a = \frac{p}{q}\ln a = r \ln a\ .$$

Derivative of The Natural Logarithm Function

مشتقة دالة اللوغاريتم الطبيعي

Theorem:

If $y = f(x) = \ln x\ \ then\ \frac{dy}{dx} = \acute{f}(x) = \frac{1}{x}$

Proof:

$$\frac{dy}{dx} = \frac{d}{dx}(\ln x) = \frac{d}{dx}\int_1^{x} \frac{dt}{t} = \frac{1}{x}\ (by fundumental\ theorem\ of\ calculus)$$

Now, If u=u(x) is a differential function of x and

$$y = \ln u \quad then \; \frac{dy}{dx} = \frac{1}{u} . \frac{du}{dx} \quad .$$

Ex. (1) Find $\frac{dy}{dx}$ of the following functions:

(1) $y = \ln(x^3 + 2x^2 - 3x + 5) \Longrightarrow \frac{dy}{dx} = \frac{3x^2+4x-3}{(x^3+2x^2-3x+5)}$.

(2) $y = \ln(x^{-2} + \sin^2 3x) \Longrightarrow \frac{dy}{dx} = \frac{(-2x^{-3}+6\sin 3x\cos 3x)}{(x^{-2}+\sin^2 3x)}$.

(3) $y = \sin^{-1}(\ln x) . \ln(\sin^{-1} 3x)$

$$\frac{dy}{dx} = \sin^{-1}(\ln x) . \frac{\frac{3}{\sqrt{1-(3x)^2}}}{\sin^{-1} 3x} + \ln(\sin^{-1} 3x) . \frac{\frac{1}{x}}{\sqrt{1-(\ln x)^2}}$$

(4) $y = \ln[\ln(\sec^2 2x + x \sin^{-1} x)]$

$$\frac{dy}{dx} = \frac{1}{\ln(\sec^2 2x + \mathrm{x}\sin^{-1} x)]} . \frac{1}{\sec^2 2x + \mathrm{x}\sin^{-1} x}$$
$$\left(2\sec 2x . \sec 2x \tan 2x . 2 + \frac{x}{\sqrt{1-x^2}} + \sin^{-1} x\right).$$

Ex. (2) If $\boldsymbol{y = \frac{x^{\frac{3}{2}}(3x+2)^{\frac{1}{2}}(x^2+3x-1)^{\frac{4}{3}}}{(x^2+2)^{\frac{5}{2}}}} \;\; \boldsymbol{find \; \frac{dy}{dx}}$

$$\ln y = \frac{3}{2}\ln x + \frac{1}{2}\ln(3x+2) + \frac{4}{3}\ln(x^2+3x-1) - \frac{5}{2}\ln(x^2+2)$$

$$\frac{1}{y}\frac{dy}{dx} = \left[\frac{3}{2}\frac{1}{x} + \frac{1}{2}\frac{3}{(3x+2)} + \frac{4}{3}\frac{(2x+3)}{(x^2+3x-1)} - \frac{5}{2}\frac{3x^2}{(x^2+2)}\right].$$

Ex. (3) If $\boldsymbol{y = \frac{(x^2+1)^{\frac{3}{2}}(x^2-1)^{\frac{3}{2}}\tan^{-1}(\sin 2x)}{(x^2+4)^{\frac{2}{3}}\sin^{-3} 2x}} \;\; \boldsymbol{find \; \frac{dy}{dx}}$

$$\ln y = \frac{3}{2}\ln(x^2+1) - \frac{3}{2}\ln(x^2-1) + \ln[\tan^{-1}(\sin 2x)] - \frac{2}{3}\ln(x^2+4) + 3\ln(\sin 2x)$$

$$\frac{1}{y}\acute{y} = [\frac{3}{2}\frac{2x}{(x^2+1)} - \frac{3}{2}\frac{2x}{(x^2-1)} + \frac{\frac{2\cos 2x}{1+\sin^2 2x}}{\tan^{-1}(\sin 2x)} - \frac{2}{3}\frac{2x}{(x^2+4)} + 6\frac{\cos 2x}{\sin 2x}]$$

Ex. (4) Find $\frac{dy}{dx}$ of the following functions:

(1) $\boldsymbol{y = x^{\sin x}}$

$$\ln y = \sin x \ln x \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\sin x}{x} + \cos x \ln x$$

$$\frac{dy}{dx} = y[\frac{\sin x}{x} + \cos x \,.\ln x] = x^{\sin x}\left[\frac{\sin x}{x} + \cos x \,.\ln x\right].$$

(2) $\boldsymbol{y = (\ln x)^x \Rightarrow \ln y = x \ln(\ln x)}$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x\frac{1}{\ln x}.\frac{1}{x} + \ln(\ln x).(1)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{\ln x} + \ln(\ln x)\right] = (\ln x)^x\left[\frac{1}{\ln x} + \ln(\ln x)\right].$$

(3) $\boldsymbol{y = (\tan^{-1} x)^{\frac{x \sin x}{x^2+1}}}$

$$\Rightarrow \ln y = \frac{x \sin x}{x^2+1}.\ln(\tan^{-1} x)$$

$$\Rightarrow \ln(\ln y) = \ln x + \ln(\sin x) - \ln(x^2+1) + \ln[\ln(\tan^{-1} x)]$$

$$\frac{1}{\ln y}.\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{\cos x}{\sin x} - \frac{2x}{(x^2+1)} + \frac{1}{\ln(\tan^{-1} x)}.\frac{1}{1+x^2}.$$

Ex. (5) Evaluate the following limits:

(1) $\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = \lim_{x\to\infty}\frac{1}{x} = \frac{1}{\infty} = 0$.

(2) $\lim_{x\to 0}\frac{\ln(1+2x)-2x}{x^2} = \lim_{x\to 0}\frac{\frac{2}{1+2x}-2}{2x} = \lim_{x\to 0}\frac{-\frac{4}{(1+2x)^2}}{2} = -2$.

(3) $\lim_{x\to\infty}\frac{\ln(\ln x)}{\ln x} = \lim_{x\to\infty}\frac{\frac{1}{\ln x}.\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{1}{\ln x} = 0$.

Ex. (6) solve for x if $\boldsymbol{3^x = 2^{x+1}}$

<u>Soul.</u>

Take logarithm of both side

$x\ln 3 = (x+1)\ln 2 \Rightarrow x\ln 3 = x\ln 2 + \ln 2$

$x\ln 3 - x\ln 2 = \ln 2 \Rightarrow x(\ln 3 - \ln 2) = \ln 2$

$$x = \frac{\ln 2}{\ln 3 - \ln 2}.$$

The Exponential Function الـــدالة الاســـــيـــة

Defn. The exponential function is defined as an inverse of natural logarithm, and denoted by exp or e.

This is:

For $-\infty < x < \infty, we\ define\ y = f(x) = e^x \Leftrightarrow x = \ln y, 0 < y < \infty$.

Properties

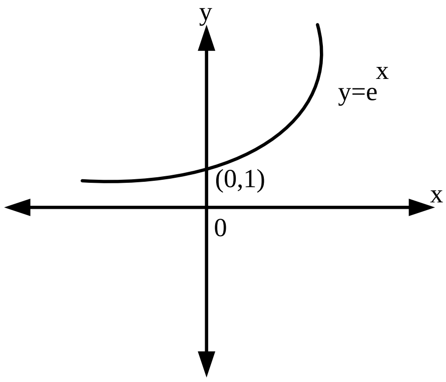
(1) e=2.7183

(2) $e^{x+y} = e^x . e^y$

(3) $e^{x-y} = \frac{e^x}{e^y}$

(4) $e^{\ln x} = x$

(5) $\ln e^x = x$



Proof (2)

Let $u = e^x \Rightarrow x = \ln u$

Let $v = e^y \Rightarrow y = \ln v$

$x + y = \ln u + \ln v \Rightarrow x + y = \ln(u.v)$

$\Rightarrow u.v = e^{x+y} \Rightarrow e^x.e^y = e^{x+y}$.

Proof (4)

Let $y = e^{\ln x} \Rightarrow \ln x = \ln y \Rightarrow x = y = e^{\ln x}$.

Proof (5)

Let $y = \ln e^x \Rightarrow e^y = e^x \Rightarrow x = y = \ln e^x$.

Ex. (1) Simplify the following expressions:

(1) $e^{\ln 2} = 2$

(2) $e^{\ln(x^2+1)} = (x^2 + 1)$

(3) $\ln e^{-1.3} = -1.3$

(4) $\ln e^{\sin x} = \sin x$

(5) $\ln(\frac{e^{2x}}{5}) = \ln e^{2x} - \ln 5 = 2x - \ln 5$

(6) $e^{\ln 2+3\ln x} = e^{\ln 2}.e^{3\ln x} = 2e^{\ln x^3} = 2x^3$

(7) $e^{2x+\ln x} = e^{2x}.e^{\ln x} = xe^{2x}$

Ex. (2) Solve for y if

(1) $\ln(y-1) - \ln y = 2x$

$$\ln(y-1) - \ln y = 2x \Rightarrow \ln(\frac{y-1}{y}) = 2x$$

$$\Rightarrow \frac{y-1}{y} = e^{2x} \Rightarrow y - 1 = ye^{2x} \Rightarrow y(1 - e^{2x}) = 1$$

$$y = \frac{1}{1 - e^{2x}}.$$

(2) $\ln(y-1) = x + \ln x$

$$\Rightarrow y - 1 = e^x \,.\, e^{\ln x}$$

$$\Rightarrow y = xe^x + 1 \;.$$

Derivative of The Exponential Function

مشتـــــقة الدالــــة الاســـــيـــــة

Theorem

If $y = e^x$ *then* $\frac{dy}{dx} = e^x$

Proof

$$y = e^x \Rightarrow x = \ln y \Rightarrow 1 = \frac{1}{y}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = y = e^x \,.$$

Now, if u(x) =u is a differentiable function of x and $y = e^u$ *then* $\frac{dy}{dx} = e^u.\frac{du}{dx}$.

Ex. (1) Find $\frac{dy}{dx}$ of the following functions:

(1) $y = e^{x^2+\sin 2x} \Rightarrow \frac{dy}{dx} = e^{x^2+\sin 2x}.(2x + 2\cos 2x)\,.$

(2) $y = e^{\tan^{-1} 2x + \ln x} = xe^{\tan^{-1} 2x} \Rightarrow \frac{dy}{dx} = xe^{\tan^{-1} 2x}.\frac{2}{1+(2x)^2} + e^{\tan^{-1} 2x}$

$$\therefore \frac{dy}{dx} = (\frac{2x}{1+4x^2} + 1)e^{\tan^{-1} 2x}$$

(3) $y = \tan^{-1}(e^{2x}) \Rightarrow \frac{dy}{dx} = \frac{2e^{2x}}{1+e^{4x}}$

(4) $y = e^{\sec x}.\sec e^x \Rightarrow \frac{dy}{dx} = e^{\sec x}.\sec(e^x)\,.\tan(e^x)\, e^x +$
$\sec(e^x)\,.\,e^{\sec x} \sec x \tan x \;.$

Ex. (2) If $y = (\sin x)^{ex}$ $find$ $\frac{de^{\sin^{-1}y}}{d\ln x}$

Let $u = e^{\sin^{-1}y}$ and $v = \ln x$

$$\frac{de^{\sin^{-1}y}}{d\ln x} = \frac{du}{dv} = \frac{du}{dy}.\frac{dy}{dx}.\frac{dx}{dv}$$

$$u = e^{\sin^{-1}y} \Longrightarrow \frac{du}{dy} = e^{\sin^{-1}y}.\frac{1}{\sqrt{1-y^2}}, v = \ln x \Longrightarrow \frac{dv}{dx} = \frac{1}{x}$$

$$y = (\sin x)^{ex} \Longrightarrow \ln y = e^x \ln(\sin x)$$

$$\Longrightarrow \frac{1}{y}\frac{dy}{dx} = [e^x.\frac{\cos x}{\sin x} + \ln(\sin x)\, e^x]$$

$$\frac{dy}{dx} = y[e^x \cot x + e^x \ln(\sin x)]$$

$$\frac{dy}{dx} = (\sin x)^{e^x}[e^x \cot x + e^x \ln(\sin x)]$$

$$\Longrightarrow \frac{\boldsymbol{de^{\sin^{-1}y}}}{\mathbf{d}\ \mathbf{ln}\, \boldsymbol{x}} = \frac{\boldsymbol{e^{\sin^{-1}y}}}{\sqrt{\mathbf{1}-\boldsymbol{y^2}}}.(\sin x)^{e^x}[e^x \cot x + e^x \ln(\sin x)].x$$

Where $y = (\sin x)^{ex}$.

Ex. (3) Evaluate the following limits:

(1) $\lim_{x\to\infty}\frac{e^x}{x^3} = \lim_{x\to\infty}\frac{e^x}{3x^2} = \lim_{x\to\infty}\frac{e^x}{6x} = \lim_{x\to\infty}\frac{e^x}{6} = \frac{e^\infty}{6} = \infty$.

(2) $\lim_{x\to\infty}\frac{x^3}{e^x} = \lim_{x\to\infty}\frac{3x^2}{e^x} = \lim_{x\to\infty}\frac{6x}{e^x} = \lim_{x\to\infty}\frac{6}{e^x} = \frac{6}{e^\infty} = \frac{6}{\infty} = 0$.

(3) $\lim_{x\to 0}\frac{e^{2x}-2x-1}{1-\cos x} = \lim_{x\to 0}\frac{2e^{2x}-2}{\sin x} = \lim_{x\to 0}\frac{4e^{2x}}{\cos x} = \frac{4e^0}{\cos x} = \frac{4}{1} = 4$.

(4) $\lim_{x\to 0} x^{\frac{1}{x}} = \lim_{x\to 0} e^{\ln x^{\frac{1}{x}}} = \lim_{x\to 0} e^{\frac{1}{x}\ln x} = \lim_{e^x\to 0}\frac{\ln x}{x}$

$= \lim_{x\to 0}\frac{\frac{1}{x}}{1} = e^{\frac{\infty}{1}} = e^\infty = \infty$.

(5) $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\ln(1+x)^{\frac{1}{x}}} = e^{\lim_{x\to 0}\frac{\ln(1+x)}{x}} = e^{\lim_{x\to 0}\frac{1}{1+x}}$

$e^{\frac{1}{1+0}} = e^1 = e = 2.7183$.

(6) $\lim_{x\to\infty}(1+\frac{b}{x})^{cx} = e^{bc}$ H.W

Ex. (4) Solve for x if $3y = \frac{e^{2x}-1}{e^{2x}+1}$ (H.W)

Ex. (5) $\lim_{x\to 4} \frac{e^{x-4}-x+4}{\cos^2(\pi x)}$ (H.W)

The Function a^x

Defn. For a>0, we define $a^x = e^{x.\ln a}$

Theorem: If $y = a^x$ *then* $\frac{dy}{dx} = a^x.\ln a$.

Proof: $y = a^x = e^{x.\ln a} \Longrightarrow \frac{dy}{dx} = e^{x.\ln a}.\ln a = a^x.\ln a$.

Now, if u=u(x) is a differentiable function of x and $y = a^u$

$$then\ \frac{dy}{dx} = a^u.\ln a.\frac{du}{dx} .$$

Ex. (1) Find $\frac{dy}{dx}$ of the following functions:

(1) $y = 2^{\sin^2 2x} \Longrightarrow \frac{dy}{dx} = 2^{\sin^2 2x}.\ln 2.(2\sin 2x\cos 2x.2)$

$= 2^{\sin^2 2x}.\ln 2.(4\sin 2x\cos 2x)$

(2) $y = 3^{\tan^{-1} 2x}.\ln(\sec 2x)$

$$\acute{y} = \frac{dy}{dx} = 3^{\tan^{-1} 2x}.\frac{2\sec 2x.\tan 2x}{\sec 2x} + \ln(\sec 2x).3^{\tan^{-1} 2x}.\ln 3.(\frac{2}{1+4x^2})$$

$$= 2.3^{\tan^{-1} 2x}\tan 2x + \ln 3.\ln(\sec 2x).3^{\tan^{-1} 2x}\left(\frac{2}{1+4x^2}\right).$$

Ex. (2) Find the following limits:

(1) $\lim_{x\to\infty} 2^{-x} = 2^{-\infty} = 0$

(2) $\lim_{x\to-\infty} 3^x = 3^{-\infty} = 0$

(3) $\lim_{x\to 0}\frac{3^{\sin x}-1}{x} = \lim_{x\to 0}\frac{3^{\sin x}.\ln 3.\cos x}{1} = 3^{º}.\ln 3.\cos 0 = (1).\ln 3.(1) = \ln 3.$

(4) $\lim_{x\to 0}\frac{3^x-2^x}{x} = \lim_{x\to 0}\frac{3^x\ln 3-2^x\ln 2}{1}$

$= 3^{º}\ln 3 - 2^{º}\ln 2 = \ln 3 - \ln 2 = \ln\left(\frac{3}{2}\right).$

Ex. (3) Solve for x if $3^{\log_3 7} + 2^{\log_2 5} = 5^{\log_5 x}$

Soul.

since $3^{\log_3 7} = e^{\log_3 7 . \ln 3} = e^{\frac{\ln 7}{\ln 3} . \ln 3} = e^{\ln 7} = 7$

And $2^{\log_2 5} = e^{\log_2 5 \ln 2} = e^{\frac{\ln 5}{\ln 2} . \ln 2} = e^{\ln 5} = 5$

So 7+5=x

$\Rightarrow$ x=12

The Hyperbolic Functions الــــدوال الــزائـــديـــــة

The hyperbolic functions are special combinations of the functions e^x *and* e^{-x}

Defn. We define,

(1) $\sinh x = \frac{e^x - e^{-x}}{2}$

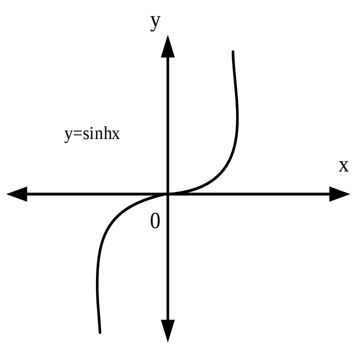
(2) $\cosh x = \frac{e^x + e^{-x}}{2}$

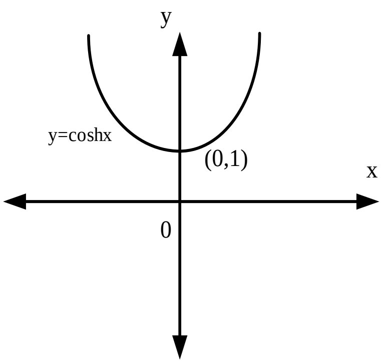
(3) $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

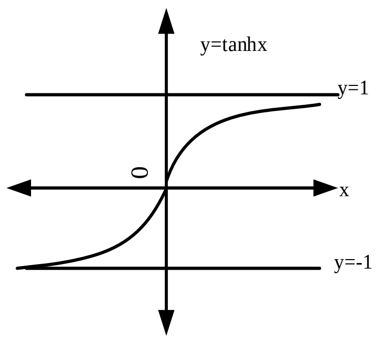
(4) $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

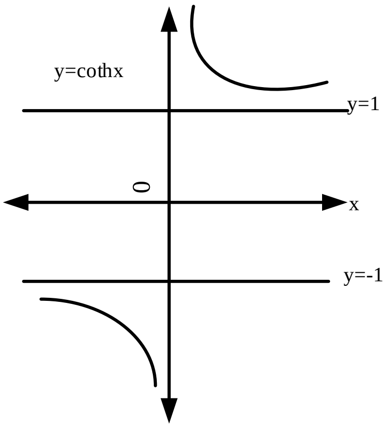
(5) $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

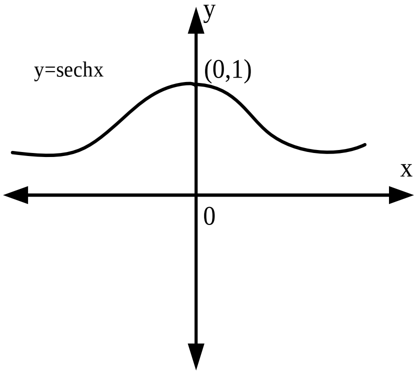
(6) $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$

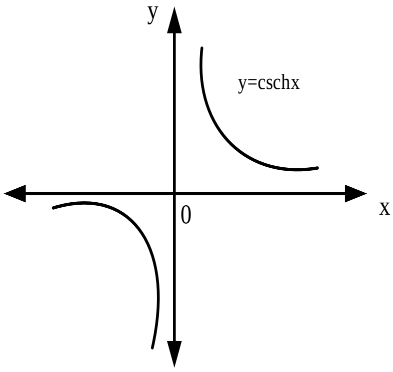












Some Important Relations and Identities

بعـــض العـــلاقات والحقـــائق الـــمهمـــة

(1) $\cosh^2 x - \sinh^2 x = 1$

(2) $\tanh^2 x + \mathrm{sech}^2\, x = 1$

(3) $\coth^2 x - \mathrm{csch}^2\, x = 1$

(4) $\sinh(-x) = -\sinh x$

(5) $\cosh(-x) = \cosh x$

(6) $\tanh(-x) = -\tanh x$

(7) $\sinh x \pm \cosh x = \pm e^{\pm x}$

(8) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

(9) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

(10) $\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

(11) $\sinh 2x = 2 \sinh x \cosh x$

(12) $\sinh^2 x = \frac{\cosh 2x - 1}{2}$

(13) $\cosh^2 x = \frac{\cosh 2x + 1}{2}$

Proof (1): $\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4}$$

$$= \frac{4}{4} = 1 \ .$$

Proof (2): $\tanh^2 x + \mathrm{sech}^2\, x = \frac{\sinh^2 x}{\cosh^2 x} + \frac{1}{\cosh^2 x} = \frac{\sinh^2 x + 1}{\cosh^2 x} = \frac{\cosh^2 x}{\cosh^2 x} = 1$.

Proof (4): $\sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = \frac{e^{x} - e^{-x}}{2} = -\sinh x$.

Proof (5): $\cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{x} + e^{-x}}{2} = \cosh x$.

Proof (6): $\tanh(-x) = \frac{\sinh(-x)}{\cosh(-x)} = \frac{-\sinh x}{\cosh x} = -\tanh x$.

Proof (7): $\sinh x + \cosh x = \frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x} + e^x + e^{-x}}{2} = \frac{2e^x}{2} = e^x$.

Ex. (1) If $\sinh x = -\frac{3}{4}$ find the value of the other hyperbolic functions:

<u>Soul.</u>

$$\cosh^2 x = 1 + \sinh^2 x = 1 + (-\frac{3}{4})^2 = 1 + \frac{9}{16} = \frac{25}{16}$$

Since $\cosh x > 0 \Longrightarrow \cosh x = \frac{5}{4}$

$$\tanh x = \frac{\sinh x}{\operatorname{csch} x} = \frac{\frac{-3}{4}}{\frac{5}{4}} = -\frac{3}{4} \quad \textit{and so on for the other hyperbolic}$$

Ex. (2) Show that $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$. (H.W)

Ex. (3) If $\sinh x = \tan\theta$ for $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$ Find the other hyperbolic functions in terms of the trigonometric. (H.W)

EX. (4) Show that: $\sinh 3x = 3\sinh x + 4\sinh^3 x$. (H.W)

Derivatives of Hyperbolic Functions:

1. If $y = f(x) = \sinh x$ $then$ $\frac{dy}{dx} = \cosh x$.
2. If $y = f(x) = \cosh x$ $then$ $\frac{dy}{dx} = \sinh x$.
3. If $y = f(x) = \tanh x$ $then$ $\frac{dy}{dx} = \operatorname{sech}^2 x$.
4. If $y = f(x) = \coth x$ $then$ $\frac{dy}{dx} = -\operatorname{csch}^2 x$.
5. If $y = f(x) = \operatorname{sech} x$ $then$ $\frac{dy}{dx} = -\operatorname{sech} x \tanh x$.
6. If $y = f(x) = \operatorname{csch} x$ $then$ $\frac{dy}{dx} = -\operatorname{csch} x \coth x$.

Proof (1) $y = \sinh x = \frac{1}{2}(e^x - e^{-x})$

$$\frac{dy}{dx} = \frac{1}{2}[e^x - e^{-x}(-1)] = \frac{1}{2}(e^x + e^{-x}) = \cosh x .$$

Proof (2) $y = \cosh x = \frac{1}{2}(e^x + e^{-x})$

$$\frac{dy}{dx} = \frac{1}{2}[e^x + e^{-x}(-1)] = \frac{1}{2}(e^x - e^{-x}) = \sinh x$$

Proof (3) $y = \tanh x = \frac{\sinh x}{\cosh x}$

$$\frac{dy}{dx} = \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$$

$$= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x \ .$$

Now, if u(x) =u is a differentiable function of x and

1. $y = \sinh u \ \ then \ \frac{dy}{dx} = \cosh u \frac{du}{dx}$
2. $y = \cosh u \ \ then \ \frac{dy}{dx} = \sinh u \frac{du}{dx}$
3. $y = \tanh u \ \ then \ \frac{dy}{dx} = \operatorname{sech}^2 u \frac{du}{dx}$
4. $y = \coth u \ \ then \ \frac{dy}{dx} = -\operatorname{csch}^2 u \frac{du}{dx}$
5. $y = \operatorname{sech} u \ \ then \ \frac{dy}{dx} = -\operatorname{sech} u \tanh u \frac{du}{dx}$
6. $y = \operatorname{csch} u \ \ then \ \frac{dy}{dx} = -\operatorname{csch} u \coth u \frac{du}{dx}$

Ex. (5) Find $\frac{dy}{dx}$ of the following functions:

(1) $y = f(x) = \sinh(x^2 + 3\sin x + \ln x)$

$$\acute{y} = \frac{dy}{dx} = \cosh(x^2 + 3\sin x + \ln x)\,.\,(2x + 3\cos x + \frac{1}{x})$$

(2) $y = f(x) = \tanh^{-2}[e^{\tan^{-1} 2x} + \operatorname{sech}^2 2x + \sin(e^{2x})]$

$$\frac{dy}{dx} = -2\tanh^{-3}[\,e^{\tan^{-1} 2x} + \operatorname{sech}^2 2x + \sin(e^{2x})].\operatorname{sech}^2[e^{\tan^{-1} 2x} + \operatorname{sech}^2 2x + \sin(e^{2x})].\,[e^{\tan^{-1} 2x}\frac{2}{1+4x} - 4\operatorname{sech}^2 2x \tanh 2x + 2\cos(e^{2x})e^{2x}]$$

(3) $y = f(x) = \operatorname{csch}^{-3}(\tan 2x + \tan^{-1} 2x)\,.\csc^{-3}(\tanh 2x)$. **(H.W)**

Ex. (6) If $y = x^{x\sinh x}$ $find\ \frac{dy^3}{dx^3}$

Let $u = y^3$ $and\ v = x^3 \Longrightarrow \frac{dy^3}{dx^3} = \frac{du}{dv} = \frac{du}{dy}.\frac{dy}{dx}.\frac{dx}{dv}$

Now, $u = y^3 \Longrightarrow \frac{du}{dy} = 3y^2$

$$v = x^3 \Longrightarrow \frac{dv}{dx} = 3x^2$$

$\ln y = x\sinh x \ln x$

$$\frac{1}{y}\frac{dy}{dx} = (x\sinh x).\frac{1}{x} + \ln x\,(x\cosh x + \sinh x)$$

$$\frac{dy}{dx} = y[\sinh x + x\ln x\cosh x + \ln x\sinh x]$$

$$\frac{dy^3}{dx^3} = \frac{du}{dv} = 3y^2.\,y[\sinh x + x\ln x\cosh x + \ln x\sinh x].\frac{1}{3x^2}$$

$where\ \ y = x^{x\sinh x}$.

Ex. (7) Evaluate the following limits:

(1) $\lim_{x\to0}\frac{\sinh x - x}{x^3} = \lim_{x\to0}\frac{\cosh x - 1}{3x^2} = \lim_{x\to0}\frac{\sinh x}{6x} = \lim_{x\to0}\frac{\cosh x}{6}$

$$= \frac{\cosh 0}{6} = \frac{1}{6} .$$

(2) $\lim_{x\to0}\frac{\tanh 2x - 2x}{3x - \sinh 3x}$ **(H.W)**

(3) $\lim_{x\to0}(\operatorname{csch} x - \coth x)$ **(H.W)**

(4) $\lim_{x\to0}\frac{x - \sinh x}{(1-\cosh x)^2}$ **(H.W)**

The Inverse of The Hyperbolic Functions معكــوس الــدوال الــزائدية

1. For $-\infty < x < \infty, we\ define\ \ y = \sinh^{-1} x\ \ if\ and\ only\ if\ x = \tanh x\ for - \infty < y < \infty$.
2. For $x \geq 1, we\ define\ y = \cosh^{-1} x\ \ iff\ x = \cosh x\ \ for\ which\ y \geq 0$
3. For $|x| < 1, we\ define\ y = \tanh^{-1} x\ \ iff\ x = \tanh y\ \ for\ which - \infty < y < \infty$.
4. For $|x| > 1, we\ define\ y = \coth^{-1} x\ \ iff\ x = \coth y = y\ for\ which\ y \neq 0$.
5. For $0 < x \leq 1, we\ define\ \ y = \operatorname{sech}^{-1} x\ \ iff\ \ x = \operatorname{sech} y\ \ for\ which\ 0 \leq y < \infty$.
6. For $x \neq 0, we\ define\ y = \operatorname{csch}^{-1} x\ \ if\ and\ only\ if\ x = \operatorname{csch} y\ \ for\ which\ y \neq 0$.

Relations

1. $\sinh^{-1} x = \operatorname{csch}^{-1} \frac{1}{x}$

2. $\cosh^{-1} x = \operatorname{sech}^{-1} \frac{1}{x}$

3. $\tanh^{-1} x = \coth^{-1} \frac{1}{x}$

Proof:

Let $y = \sinh^{-1} x \Longrightarrow x = \sinh y = \frac{1}{\operatorname{csch} y} \Rightarrow \operatorname{csch} y = \frac{1}{x}$

$$\Rightarrow y = \operatorname{csch}^{-1} \frac{1}{x} \Longrightarrow \sinh^{-1} x = \operatorname{csch}^{-1} \left(\frac{1}{x}\right).$$

Expressions for the inverse of the hyperbolic functions

1. $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$

2. $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$

3. $\tanh^{-1} x = \frac{1}{2} \ln(\frac{1+x}{1-x})$

4. $\coth^{-1} x = \frac{1}{2} \ln(\frac{x+1}{x-1})$

5. $\operatorname{sech}^{-1} x = \ln \left(\frac{1+\sqrt{1-x^2}}{x}\right)$

6. $\operatorname{csch}^{-1} x = \ln \left(\frac{1+\sqrt{1+x^2}}{x}\right)$

Proof (1):

Let $y = \sinh^{-1} x \Longrightarrow x = \sinh y = \frac{e^y - e^{-y}}{2} \Longrightarrow 2x = \frac{e^y - e^{-y}}{1}$

$$\Longrightarrow 2x = e^y - \frac{1}{e^y} \Longrightarrow 2xe^y = e^{2y} - 1 \Longrightarrow (e^y)^2 - 2xe^y - 1 = 0$$

Let $z = e^y \Longrightarrow z^2 - 2xz - 1 = 0 \Longrightarrow z = \frac{2x \pm \sqrt{4x^2 - 4(-1)(1)}}{2}$

$$z = \frac{2x \pm 2\sqrt{x^2 + 1}}{2} = x \pm \sqrt{x^2 + 1} = e^y$$

since $e^y > 0 \Longrightarrow x - \sqrt{x^2 + 1} > 0 \Longrightarrow x > \sqrt{x^2 + 1}$ *impossible*

$$e^y = x + \sqrt{x^2 + 1} \Longrightarrow y = \ln\left(x + \sqrt{x^2 + 1}\right).$$

Proof (3):

$$let\ y=\tanh^{-1}x \Longrightarrow x=\tanh y=\frac{e^{y}-e^{-y}}{e^{y}+e^{-y}}=\frac{e^{y}-\frac{1}{e^{y}}}{e^{y}+\frac{1}{e^{y}}}$$

$$x=\frac{e^{2y}-1}{e^{2y}+1} \Longrightarrow xe^{2y}+x=e^{2y}-1$$

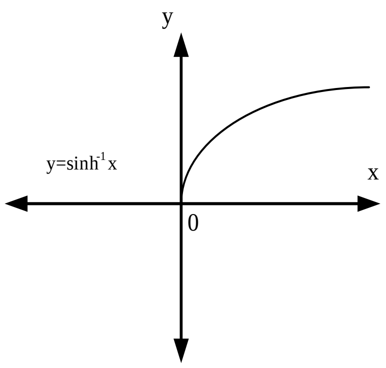
$$\Longrightarrow 1+x=e^{2y}(1-x) \Longrightarrow e^{2y}=\frac{1+x}{1-x} \Longrightarrow$$

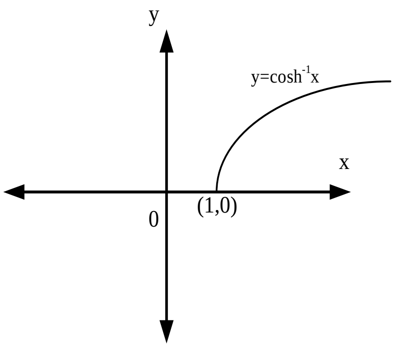
$$2y=\ln\left(\frac{1+x}{1-x}\right) \Longrightarrow y=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

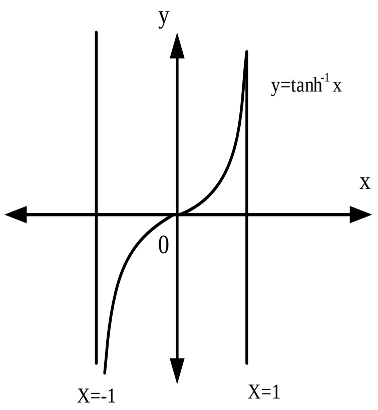
Note $\sinh^{-1}(\sinh x)=x\ ,\sinh(\sinh^{-1}x)=x$

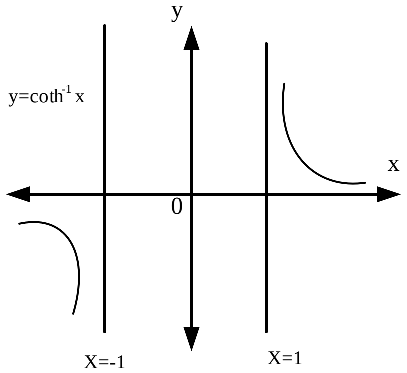
Similarly for the other hyperbolic functions.

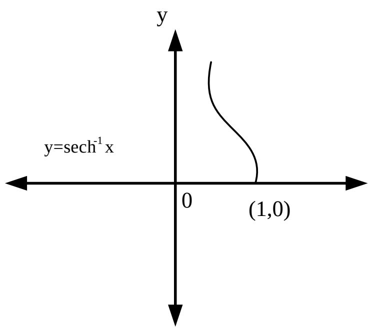
Graph of The Inverse of Hyperbolic Functions

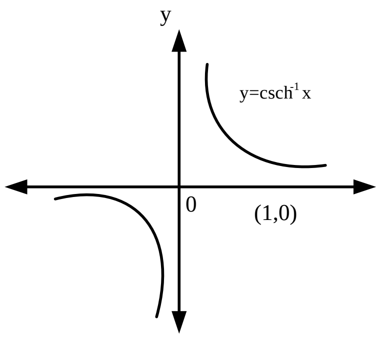










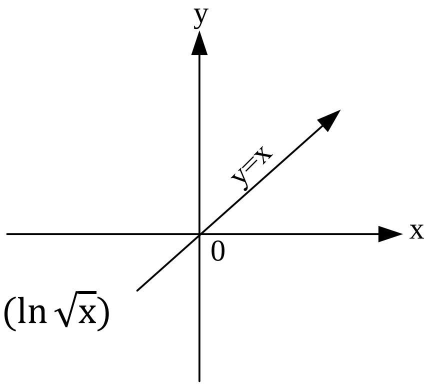


Ex. (8) Evaluate $\tan h^{-1}\left(-\frac{1}{2}\right) = \frac{1}{2}\ln\left(\frac{1+\left(-\frac{1}{2}\right)}{1-\left(-\frac{1}{2}\right)}\right) = \frac{1}{2}\ln\left(\frac{1/2}{3/2}\right)$

$$= \frac{1}{2}(\ln(1) - (\ln(3)) = -\frac{1}{2}\ln 3\,.$$

Ex. (9) Sketch the graph of $y = \frac{1}{2}\ln\left(\frac{1+\tanh x}{1-\tanh x}\right) = \tan h^{-1}\ (\tanh x) = x$

$$\therefore y = x$$



Ex. (10) Find the asymptotes of the function $y = \tan h\,(\ln\sqrt{x})$

Solu. $y = \frac{e^{2\ln\sqrt{x}}-1}{e^{2\ln\sqrt{x}}+1} = \frac{e^{\ln x}-1}{e^{\ln x}+1} = \frac{x-1}{x+1}$

$x = -1$ is V. Asymp.

$y = 1$ is H. Asymp.

Derivatives:

1. If $y = \sin h^{-1}x$ then $\acute{y} = \frac{1}{\sqrt{x^2+1}}$
2. If $y = \cos h^{-1}x$ then $\acute{y} = \frac{1}{\sqrt{x^2-1}}$
3. If $y = \begin{matrix}\tan h^{-1}x \\ \cot h^{-1}x\end{matrix}$ then $\acute{y} = \frac{1}{1-x^2}$
4. If $y = \sec h^{-1}x$ then $\acute{y} = -\frac{1}{x\sqrt{1-x^2}}$
5. If $y = \csc h^{-1}x$ then $\acute{y} = -\frac{1}{x\sqrt{1+x^2}}$

Prof (1) $y = \sin h^{-1}x \Rightarrow x = \sin hy \Rightarrow 1 = \cos hy\,\frac{dy}{dx}$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos hy} = \frac{1}{\sqrt{\sin h^2 y + 1}} = \frac{1}{\sqrt{x^2+1}}\,.$$

Now, if $u = u(x)$ is a differentiable function of x and

1. If $y = \sinh^{-1}u$ then $\frac{dy}{dx} = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$
2. If $y = \cosh^{-1}u$ then $\frac{dy}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$
3. If $y = \begin{matrix}\tanh^{-1}u \\ \coth^{-1}u\end{matrix}$ then $\frac{dy}{dx}\ \frac{1}{1-u^2}\frac{du}{dx}$
4. If $y = \operatorname{sech}^{-1}u$ then $\frac{dy}{dx} = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$
5. If $y = \operatorname{csch}^{-1}x$ then $\frac{dy}{dx} = -\frac{1}{u\sqrt{1+u^2}}\frac{du}{dx}$

<u>Ex. (11)</u> Find $\frac{dy}{dx}$ of the following functions

1. $y = \sinh^{-1}(x^2 + \sin^2 x) \Rightarrow \acute{y} = \frac{2x+2\sin x\cos x}{\sqrt{(x^2+\sin^2 x)^2+1}}$
2. $y = \operatorname{sech}^{-3}[\operatorname{sech}^{-2}(\operatorname{sech}^{-1} 2x)]$ **(H.W.)**

Exercises to solve:

[1] Simplify the expressions: بســـط التعـــابير الاتــيـة

1. $\log_2\left(e^{(\ln 2)(\sin x)}\right)$
2. $\log_4\left(2^{e^x \sin x}\right)$
3. $\log_2(e^x)$
4. $\log_9 3x$
5. $25^{\log_5(3x^2)}$
6. $\frac{\log_9 x}{\log_3 x}$
7. $\frac{\log_x a}{\log_{x^2} a}$
8. $\frac{\log_{\sqrt{10}} x}{\log_{\sqrt{2}} x}$

[2] Solve the equations: حـــل المعـــادلات الاتـــيـة

1. $8^{\log_8(3)} - e^{\ln 5} = x^2 - 7^{\log_7(3x)}$
2. $3^{\log_3(x^2)} = 5e^{\ln x} - 3\ .\ 10^{\log_{10}(2)}$
3. $\ln e + 4^{-2\log_4(x)} = \frac{1}{x}\log_{10}(100)$

[3] Find $\frac{dy}{dx}$ اوجـــد المشتـــــقة

1. $y = (x+1)^x$ **2.** $y = x^{(x+1)}$ **3.** $y = x^{\ln x}$

4. $y = (\ln x)^{\ln x}$ **5.** $y = x\log_3 (e^{(\sin x)\ln(3)})$

6. $y = 3\log_8(\log_2 x)$ **7.** $y = \log_7\left(\frac{\sin x \cos x}{e^x \; 2^x}\right)$

8. $y = \log_2\left(\frac{x^2 \; e^2}{2\sqrt{x+1}}\right)$ **9.** $y = \log_5\left(\sqrt{\left(\frac{7x}{3x+2}\right)^{\ln 5}}\right)$

Solutions الحــــلــــول

[1] Simplify the expressions:

1. $\log_2\left(e^{(\ln 2)(\sin x)}\right) = \frac{\ln(e^{\ln 2 \sin x})}{\ln 2} = \frac{\ln 2 \sin x}{\ln 2} = \sin x$

2. $\log_4\left(2^{e^x \sin x}\right) = \frac{\ln\left(2^{e^x \sin x}\right)}{\ln 4} = \frac{e^x \sin x \ln 2}{2\ln 2}$

$= \frac{1}{2} e^x \sin x$

3. $\log_e(e^x) = \frac{\ln e^x}{\ln e} = \frac{x}{1} = x$

4. $\log_9 3x = \frac{\ln 3x}{\ln 9} = \frac{\ln(3x)}{2\ln 3} = \frac{\ln 3 + \ln x}{2\ln 3}$

5. $25^{\log_5(3x^2)} = e^{\log_5(3x^2).\ln(25)} = e^{\frac{\ln 3x^2}{\ln 5}.2\ln 5} = e^{2\ln 3x^2} = e^{\ln(3x^2)^2} = 9x^4$

6. $\frac{\log_9 x}{\log_3 x} = \frac{\frac{\ln x}{\ln 9}}{\frac{\ln x}{\ln 3}} = \frac{\frac{\ln x}{2\ln 3}}{\frac{\ln x}{\ln 3}} = \frac{1}{2}$

7. $\frac{\log_x a}{\log_{x^2} a} = \frac{\frac{\ln a}{\ln x}}{\frac{\ln a}{\ln x^2}} = \frac{\frac{\ln a}{\ln x}}{\frac{\ln a}{2\ln x}} = 2$

8. $\frac{\log_{\sqrt{10}} x}{\log_{\sqrt{2}} x} = \frac{\frac{\ln x}{\ln\sqrt{10}}}{\frac{\ln x}{\ln\sqrt{2}}} = \frac{\ln\sqrt{2}}{\ln\sqrt{10}} = \frac{\ln 2^{1/2}}{\ln 10^{1/2}} = \frac{\frac{1}{2}\ln 2}{\frac{1}{2}\ln 10} = \frac{\ln 2}{\ln 10}$

[2] Solve the equations:

1. $8^{\log_8(3)} - e^{\ln 5} = x^2 - 7^{\log_7(3x)}$

$e^{\log_8(3).\ln 8} - 5 = x^2 - e^{\log_7(3x).\ln 7}$

$e^{\frac{\ln 3}{\ln 8}.\ln 8} - 5 = x^2 - e^{\frac{\ln(3x)}{\ln 7}.\ln 7}$

$$e^{\ln 3} - 5 = x^2 - e^{\ln 3x} \Rightarrow 3 - 5 = x^2 - 3x$$

$$\Rightarrow x^2 - 3x + 2 = 0 \Rightarrow (x-2)(x-1) = 0$$

$$x = 2 \quad x = 1$$

2. $3^{\log_3 (x^2)} = 5e^{\ln x} - 3 \,.\, 10^{\log_{10} (2)}$

$$e^{\log_3 (x^2).\ln 3} = 5x - 3\; 10^{\log_{10} (2).\ln 10}$$

$$e^{\frac{\ln x^2}{\ln 3}.\ln 3} = 5x - 3\; e^{\frac{\ln 2}{\ln 10}.\ln 10}$$

$$x^2 = 5x - 3\,(2) \Rightarrow x^2 - 5x + 6 = 0$$

$$(x-3)(x-2) = 0$$

$$x = 3 \quad x = 2$$

3. $\ln e + 3^{-2\log_4 (x)} = \frac{1}{x}\log_{10} (100)$

$$1 + e^{-2\log_4 (x).\ln 4} = \frac{1}{x}\,\frac{\ln 100}{\ln 10}$$

$$1 + e^{\frac{-2\ln x}{\ln 4}.\ln 4} = \frac{1}{x}\,\frac{2\ln 10}{\ln 10}$$

$$1 + e^{\ln x^{-2}} = \frac{1}{x}(2) \quad \Rightarrow \quad 1 + \frac{1}{x^2} = \frac{2}{x}$$

$$\Rightarrow \frac{x^2+1}{x^2} = \frac{2}{x} \Rightarrow 2x^2 = x^3 + x$$

$$\Rightarrow x^3 + x - 2x^2 = 0 \quad \Rightarrow \quad x^3 - 2x^2 + x = 0 \quad \Rightarrow \quad x(x^2 - 2x + 1) = 0$$

$$\Rightarrow x = 0 \qquad \Rightarrow x^2 - 2x + 1 = 0$$

$$(x-1)(x-1) = 0 \quad \Rightarrow x = 1$$

[4] Find $\frac{dy}{dx}$

5. $y = x\log_3 \left(e^{(\sin x)\ln 3}\right) = x\,\frac{\ln\left(e^{\sin x\,\ln 3}\right)}{\ln 3} = x\,\frac{\sin x\;\ln 3}{\ln 3}$

$$\therefore y = x\sin x \quad \Rightarrow \frac{dy}{dx} = x\cos x + \sin x\,(1).$$

$$\textbf{6. } y = 3\log_8\left(\log_2 x\right) = \frac{3\ln\left(\log_2 x\right)}{\ln 8} = \frac{3\ln\left(\frac{\ln x}{\ln 2}\right)}{\ln 8}$$

$$y = 3\,\frac{[\ln(\ln x) - \ln(\ln 2)]}{3\ln 2} = \frac{1}{\ln 2}\,[\ln(\ln x) - \ln(\ln 2)]$$

$$\frac{dy}{dx} = \frac{1}{\ln 2}\left[\frac{\frac{1}{x}}{\ln x} - 0\right] = \frac{1}{\ln 2(x\ln x)}\ .$$

$$\textbf{7. } y = \log_7\left(\frac{\sin x\ \ \cos x}{e^x\ \ 2^x}\right) = \frac{\ln\left(\frac{\sin x\ \cos x}{e^x\ 2^x}\right)}{\ln 7}$$

$$y = \frac{1}{\ln 7}\,[\ln(\sin x\cos x) - \ln(e^x\ \ 2^x)]$$

$$y = \frac{1}{\ln 7}\,[\ln(\sin x) + \ln(\cos x) - (\ln e^x + \ln 2^x)]$$

$$y = \frac{1}{\ln 7}[\ln(\sin x) + \ln(\cos x) - x - x\ln 2]$$

$$\therefore \frac{dy}{dx} = \frac{1}{\ln 7}\left[\frac{\cos x}{\sin x} + \frac{-\sin x}{\cos x} - 1 - \ln 2\right]\ .$$

$$= \frac{1}{\ln 7}[\cot x - \tan x - 1 - \ln 2]$$

$$\textbf{8. } y = \log_2\left(\frac{x^2e^2}{2\sqrt{x+1}}\right) = \frac{\ln\left(\frac{x^2e^2}{2\sqrt{x+1}}\right)}{\ln 2}$$

$$y = \frac{1}{\ln 2}\left[\ln(x^2e^2) - \ln\left(2\sqrt{x+1}\right)\right]$$

$$y = \frac{1}{\ln 2}\left[2\ln x + 2\ln e - \left(\ln 2 + \frac{1}{2}\ln(x+1)\right)\right]$$

$$y = \frac{1}{\ln 2}\left[2\ln x + 2 - \ \ln 2 - \frac{1}{2}\ln(x+1)\right]$$

$$\frac{dy}{dx} = \frac{1}{\ln 2}\left[\frac{2}{x} + 0 - 0 - \frac{1}{2}\ \frac{1}{x+1}\right]$$

$$\frac{dy}{dx} = \frac{2}{x\ln 2} - \frac{1}{2\ln 2\ (x+1)}$$

9. $y = \log_5\left(\sqrt{\left(\frac{7x}{3x+2}\right)^{\ln 5}}\right)$

$$y = \frac{\ln\sqrt{\left(\frac{7x}{3x+2}\right)^{\ln 5}}}{\ln 5} = \frac{1}{\ln 5}\left[\frac{1}{2}\ln\left(\frac{7x}{3x+2}\right)^{\ln 5}\right]$$

$$= \frac{1}{\ln 5}\left[\frac{\ln 5}{2}\ln\left(\frac{7x}{3x+2}\right)\right]$$

$$= \frac{1}{2}[\ln(7x) - \ln(3x+2)]$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}.\left[\frac{7}{7x} - \frac{3}{3x+2}\right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x} - \frac{3}{3x+2}\right].$$

Exercises to solve:

[1] Find the domain and the range of the following functions:

1. $y = f(x) = \sqrt{4 - x^2}$

2. $y = \sqrt{x^2 - 16}$

3. $y = \frac{1}{x-2}$

4. $y = \frac{1}{x^2-9}$

5. $y = \frac{x}{x^2+4}$

6. $y = \frac{x-1}{x^2+2}$

7. $y = x^2 + 4$

8. $y = \frac{2x}{(x-2)(x+1)}$

9. $y = \frac{x^2-1}{x^2+1}$

10. $y = \frac{1}{\sqrt{9 - x^2}}$

11. $y = \sqrt{x^2 + 4}$

12. $y = \frac{x}{x+3}$

13. $y = \sqrt{x - 4}$

14. $y = \sqrt{\frac{x}{2-x}}$

[2] Find:

1. $\lim_{x\to 2} \frac{4-x^2}{3-\sqrt{x^2+5}}$ **2.** $\lim_{x\to\infty} \frac{3x-2}{9x+7}$

3. $\lim_{x\to 2} \frac{x-2}{\sqrt{x^2-4}}$ **4.** $\lim_{x\to 1} \frac{x-1}{x^2-1}$

5. $\lim_{x\to 2} \frac{x^2-4}{x^2-5x+6}$ **6.** $\lim_{x\to -1} \frac{x^2+3x+2}{x^2+4x+3}$

7. $\lim_{h\to 0} \frac{(x+h)^2-x^2}{h}$ **8.** $\lim_{x\to 1} \frac{x-1}{\sqrt{x^2+3}-2}$

[3] Evaluate:

1. $\lim_{x\to\infty} \frac{x+3}{x^2+5x+6}$ **2.** $\lim_{x\to\infty} \frac{x}{x^2+5}$

3. $\lim_{x\to\infty} \frac{2x+3}{4x-5}$ **4.** $\lim_{x\to\infty} \frac{2x^2+1}{6+x-3x^2}$

5. $\lim_{x\to\infty} \frac{x^2+5x+6}{x+1}$

[4] Let $f(x) = x^2 - 3x$ **, Find** $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

[5] Let $f(x) = \sqrt{5x+1}$**, Find** $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

When $x > -\frac{1}{5}$.

[6] If $f(x) = \frac{x-1}{x+1}$ **, Prove that 1.** $f\left(\frac{1}{x}\right) = -f(x)$

2. $f\left(-\frac{1}{x}\right) = -\frac{1}{f(x)}$

[7] If $f(x) = x^2 - x$ **, Prove that** $f(1+x) = f(-x)$

[8] If $f(x) = \frac{1}{x}$ **, Prove** $f(a) - f(b) = f\left(\frac{ab}{b-a}\right)$

[9] If $y = f(x) = \frac{(5x+3)}{(4x-5)}$**, Prove that** $x = f(y)$.

Exercises to solve:

[1] Evaluate the following limits: (if there exist)

1. $\lim\limits_{x\to 0} \frac{\sin^2 2x}{x\ \sin^2 3x}$

2. $\lim\limits_{x\to 0} \frac{\sin ax}{\sin bx}$

3. $\lim\limits_{x\to 0} \frac{\sin x\,(1-\cos x)}{x^2}$

4. $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2}$

5. $\lim\limits_{x\to 0} \frac{1-\cos x}{\sin x}$

6. $\lim\limits_{x\to 0} x \cot x$

7. $\lim\limits_{x\to 2} \sin\left(\frac{1}{x}-\frac{1}{2}\right)$

8. $\lim\limits_{x\to 0}(\sec x + \tan x)$

9. $\lim\limits_{x\to \pi} \cos^2 x$

10. $\lim\limits_{x\to 0} \frac{\sin(a+x)-\sin a}{x}$

11. $\lim\limits_{x\to 0} x \csc x$

12. $\lim\limits_{x\to \frac{\pi}{4}} \frac{\sin x}{\cos x}$

13. $\lim\limits_{x\to 0} \frac{\cos(a+x)-\cos a}{x}$

14. $\lim\limits_{x\to \pi} \sec(1+\cos x)$

15. $\lim\limits_{x\to \infty} \sin\left(\frac{\pi x}{2-3x}\right)$

16. $\lim\limits_{x\to 0} \frac{x}{\cos\left(\frac{\pi}{2}-x\right)}$

17. $\lim\limits_{x\to \frac{\pi}{2}} \frac{\sin\left(x-\frac{\pi}{2}\right)}{\left(\frac{\pi}{2}-x\right)}$

18. $\lim\limits_{x\to \frac{-\pi}{2}} \frac{\sin\left(\frac{\pi}{2}+x\right)}{\left(x+\frac{\pi}{2}\right)}$

قـــوانــــيـــن مــهـــمـــة

1. $\sin\left(x-\frac{\pi}{2}\right) = -\cos x$
2. $\sin\left(x+\frac{\pi}{2}\right) = \cos x$
3. $\cos\left(x-\frac{\pi}{2}\right) = \sin x$
4. $\cos\left(x+\frac{\pi}{2}\right) = -\sin x$
5. $\sin(\pi - x) = \sin x$
6. $\cos(\pi - x) = -\cos x$

[2] Shows the asymptotes of

1. $y = \frac{x}{x-1}$

2. $y = \frac{8}{2-x^2}$

3. $y = \frac{1}{2x-3}$

4. $y = \frac{x^2-4}{x^2-1}$

5. $y = \frac{x^2+1}{x^2-4x+3}$

6. $y = \frac{x^2+8}{x^2-4}$

7. $y = \frac{x-4}{x-5}$

8. $y = x - \frac{1}{x}$

9. $y = \frac{x}{2} + \frac{2}{x}$

10. $y = x^2 + \frac{1}{x^2}$

11. $y = \frac{x^2+1}{x^3-4x}$

12. $y = \frac{x^3-4x}{x^3-x}$

Exercises to solve:

[1] If $\acute{f}(x) = \sin(x^2)$ and $y = f\left((2x-1)/(x+1)\right)$. Find $\frac{dy}{dx}$

[2] If $\acute{f}(x) = \sqrt{3x^2-1}$ and $y = f(x^2)$. Find $\frac{dy}{dx}$.

[3] Given. $y = 3\sin 2x$ and $x = u^2 + \pi$, Find the value of $\frac{dy}{dx}$ when $u = 0$

[4] Find $\frac{dy}{dx}$ if $x = \cos 3t$ and $y = \sin^2 3t$

[5] If $y = x^2 + 1$ and $u = \sqrt{x^2+1}$, Find $\frac{dy}{du}$

[6] If $x = y^2 + y$ and $u = (x^2+x)^{\frac{3}{2}}$ Find $\frac{dy}{du}$

[7] If $x = t - t^2$ and $y = t - t^3$, Find the value of $\frac{dy}{dx}$ at $t = 1$

[8] Find the value of

$$\lim_{\Delta x \to 0} \frac{[2-3(x+\Delta x)]^2 - [2-3x]^2}{\Delta x}, \Delta x \neq 0$$

[9] Find

1. $\lim_{x\to\infty} \frac{x}{x-1}$

2. $\lim_{s\to\infty} \left(\frac{s}{s-1}\right)\left(\frac{s^2}{5+s^2}\right)$

3. $\lim_{x\to\infty} \left(\frac{1}{x}+1\right)\left(\frac{5x^2-1}{x^2}\right)$

4. $\lim_{x\to\infty} \left(1+\cos\frac{1}{x}\right)$

5. $\lim_{x\to\infty} \frac{\sin 2x}{x}$

6. $\lim_{x\to\infty} \frac{x+\sin x}{x+\cos x}$

7. $\lim_{x\to 0} \frac{|x|}{|x|+1}$

8. $\lim_{x\to -5} \frac{x^2+3x-10}{x+5}$

[10] Find the domain and the range of the functions

1. $y = \sqrt{\frac{1}{x}-1}$

2. $y = \frac{x}{x-2}$

3. $y = 1+\frac{1}{x}$

4. $y = \frac{1}{x-2}$

5. $y = \frac{1}{x+1}$

6. $y = \frac{x}{x+1}$

7. $y = \sin^{-1}(5x-1)$

8. $y = \cos^{-1}(3x+2)$

9. $y = \sin^{-1}(2x+1)$

10. $y = \sec^{-1}(3x+2)$

11. $y = \csc^{-1}(3x+2)$

12. $y = \sec^{-1}(3x+3)$

13. $y = \csc^{-1}(2x+1)$

14. $y = \sin x$

15. $y = \cos x$

Exercises to solve:

[1] Find $\frac{dy}{dx}$ of the following functions:

1. $y = \cosh^{-1}(x^2+\sin^3 x)$

2. $y = \sinh^{-1}(x^2+\sin^2 x)$

3. $y = 2^x \operatorname{sech} 2x$

4. $y = x^3 . e^{-x^2} . \cosh^3 x . \sinh^{-1} x$

5. $y = 3^{\tanh^{-1} 2x} . \ln(\sec 2x)$

6. $y = (\tan^{-1} x)^{\frac{x \sinh x}{x^2+1}}$

[2] Find the asymptotes of the following functions

1. $y = \tan h\left(\ln\sqrt{x}\right)$ **2.** $y = \cot h\left(\ln\sqrt{x}\right)$

[3] If $y = (\sin h^{-1})^{e^x}$. find $\frac{de^{\sin^{-1} y}}{d \ln x}$

[4] If $y = (\sin h^{-1})^{\ln x}$. find $\frac{de^{\sin h^{-1} y}}{de^x}$

[5] Find $\frac{dy}{dx}$ of the following function:

$$y = \frac{(x^2+1)^{\frac{3}{2}}.(x^2-1)^{-\frac{3}{2}}.\tan h^{-1}\ (\sin h2x)}{(x^2+4)^{\frac{3}{2}}.\sin h^{-3}\ (2x)}$$

[6] Prove that: $\cos h2x = \cos h^2 x + \sin h^2 x$

$$= 2\sin h^2 x + 1$$

$$= 2\cos h^2 x - 1$$

[7] Prove that: $\tan h2x = \frac{2\tan hx}{\tan h^2 x}$

[8] If $y = a\cos h\frac{x}{a}$ then $\frac{d^2y}{dx^2} = \frac{1}{a}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ (prove that)

[9] Prove that: if $y = A\cos h\ bx + B\sin h\, bx$

Where A, B, b are Constants, then $\frac{d^2y}{dx^2} = b^2 y$.

[10] Find $\frac{dy}{dx}$ of the following functions:

1. $y = \tan h^{-1}\ (\sin x)$

2. $x = a\sec h^{-1}\left(\frac{y}{a}\right) - \sqrt{a^2 - y^2}$. where a: constant

3. $y = 2\tan h^{-1}\left(\tan\frac{x}{2}\right)$

4. $y = \tan h\ (1 + x^2)$

5. $y = x\sec h\ x^2$

6. $y = \csc h^2\ (x^2 + 1)$

7. $y = \frac{1}{4}\sin h\ 2x - \frac{1}{2}x$

8. $y = \ln(\tan h 2x)$

9. $y = \ln(\tan h^{-1} 2x)$

10. $y = e^{\tan h 2x}$

[11] Prove that: if $x = \sec h^{-1} y - \sqrt{1 - y^2}$, then $\frac{dy}{dx} = \frac{-y}{\sqrt{1-y^2}}$.

Complex Numbers (الاعـــداد المـعقــدة (المـــركــبـــة

Number Systems

1. **<u>Natural number and integers:</u>** The counting numbers 1, 2, 3, …. are called the natural number (الاعداد الطبيعية) or the positive integers. The number zero and the negative integers are added to the positive integers to from the system of all integers

   …, -3, -2, -1, 0, 1, 2, 3, …

   This system of numbers is closed under the operations of addition and multiplication. That is, if m and n are any integers, then

   $$m + n = p \quad and \quad mn = q$$

   Are also integers

   In the system of all integers, we can solve equations of the form

   $x + a = 0,$ ... (1)

   Where a may be any integer.

2. **<u>Rational numbers (الاعـــداد النسبية)</u>**: The second system introduces fraction: which are just ordered pairs m/n of integers m and n. We can have all rations of integers, excluding those having zero in the denominator. This system is called the set of

rational numbers. By this system the rational operations of arithmetic may be performed they are

In the system of all rational numbers, we can solve all equations of the form

$$ax + b = 0 \quad .... (2)$$

Provided a and b are rational numbers and $a \neq 0$.

3. **Irrational numbers:** some number, such as $\sqrt{2}$, cannot be expressed as the ratio of two integers. Multiples of some other, more fundamental unit. That is we cannot find a rational number solution of the eq. $x^2 = 2$, because these is no rational number whose square is 2. Numbers such as $\sqrt{2}$ are called irrational numbers.

   If we add to the system above, the system of rational numbers we arrive at the system of all real numbers.

   In the system of all real numbers, we can solve all the eqs. in (1) and (2) and all quadratic equations

   $ax^2 + bx + c = 0$

   Having $a \neq 0 \;\; and \;\; b^2 - 4\,ac \geq 0$

4. **Complex numbers:**

   The solution of the quadratic equations

   $x^2 + 1 = 0 \quad or \quad ax^2 + bx + c = 0 \;\; when \;\; b^2 - 4\,ac < 0$

   Involves the use of the system of complex numbers $a + ib$

   The symbol i is then defined as $i = \sqrt{-1}$ or $i^2 = -1$ we may use a notation such as (a, b). Then it might be said that the complex number system consists of all ordered pairs of real numbers (a, b), with the rules by which they are to be equated added, multiplied, and so on. We shall use both the (a, b) notation and the notation. $a + ib$. We call a the real part, and b the imaginary part of (a, b).

Complex Number العــدد المــعــقـــد

يعرف العدد z بأنه زوج مرتب z = (x, y) حيث ان y, x عددان حقيقيان و الخاضعات لعمليتي الجمع والضرب وكما يلي:-

اذا كان $z_1 = (x_1, y_1)$ ، $z_2 = (x_2, y_2)$ فأن

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$z_1 . z_2 = (x_1, y_1).(x_2, y_2) = (x_1x_2 - y_1y_2 , x_1y_2 + x_2y_1)$$

الازواج المرتبة بالصيغة (0,y) تسمى بالاعداد الخيالية الصرفة (Pure imaginary numbers) والازواج المتربة بالصيغة x ويرمز لمجموعة الاعداد المعقدة ℂ.

ويسمى x بالجزء الحقيقي للعدد المعقد z=(x, y) ويرمز له Re (z)

ويسمى y بالجزء الحقيقي للعدد المعقد z=(x, y) ويرمز له Im (z)

اي ان **x= Re (z)** و **y= Im (z)**

مـــــلاحـــظـــة

الازواج المرتبة (0, 0) هو العدد الحقيقي صفر

يمكن كتابة عدد معقد z=(x, y) بالصيغة

z=(x, y) = (x, 0) + (0, y)

=(x, 0) + (0, 1) . (y, 0)

= x+ iy

حيث ان i = (0, 1)

مـــــلاحـــظـــة

$i^2 = -1$

Example:

The complex number z= (2, 3) can be written as z= 2+ i3

Where Re (z) =2 and Im(z) =3

By using the multiplication rule of complex number, we find

$i^2 = i.i = (0,1).(0,1) = (-1,0) = 1$

$i^3 = i^2.i\ (-1)i = \ -i$

$i^4 = i^2.i^2 = (-1)(-1) = 1$

In general i^n (where n is a positive integer number) equal 1, $i, -1, -i$ when n=0, 1, 2, 3 respectively

Properties

1. يكون العدد المعقد z مساوياً للصفر اذا وفقط اذا كان كل من جزيئة الحقيقي والخيالي صفراً

$z = x + iy = 0 \Leftrightarrow x = 0 \quad and \quad y = 0$

2. يتساوى العددان المعقدان اذا وفقط اذا تساوى جزيئها الحقيقان وتساوى جزيئهما الخياليان.

أي ان

$$x_1 + iy_1 = x_2 + iy_2 \Leftrightarrow x_1 = x_2 \quad and \quad y_1 = y_2$$

العمـلـيات الأربعـــة على الاعــداد المعـقـــدة (المــركـــبـــة)

تتم عمليات الجمع وطرح وقسمة وضرب الاعداد المعقدة بأتباع قواعد الجمع والطرح والضرب والقسمة للاعداد الحقيقية مع ملاحظة $(i^2 = -1)$ على النحو الاتي:

$z_1 = x_1 + iy_1\ , \quad z_2 = x_2 + iy_2$

$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

$z_1.z_2 = (x_1 + iy_1).(x_2 + iy_2\ )$

$\qquad = (x_1x_2 - y_1y_2\ ) + i(x_1y_2 + x_2y_1\ )$

حاصل قسمة العدد z_1 على العدد $z_2(z_2 \neq 0)$ يكتب $\frac{z_1}{z_2}$ هو العدد z بحيث يحقق العلاقة

$z_1 = zz_2$

$x_1 + iy_1 = (x + iy)(x_2 + iy_2)$

وبمساواة الجزئيين الحقيقي والخيالي وبعد فتح القوسين ينتج

$x_1 = xx_2 - yy_2$

$y_1 = xy_2 + yx_2$

وبحل هاتين المعادلتين للحصول x على y و نجد ان

$$x = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} \quad and \quad y = \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$$

اذاً حاصل القسمة $\frac{z_1}{z_2}$ $(z_1 \neq 0)$ هو $\frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}$

$$= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$$

Complex Conjugate مـــرافـــق العـــدد المعـقـــد

ان مرافق العدد $z = x + iy$ ويرمز له بالرمز $\bar{z}$ هو العدد المعقد $\bar{z} = x - iy$ اي ان المرافق العدد المعقد ينتج بتغير أشارة الجزء الخيالي منه

Example:

The conjugate number $z = 2 - 3i$ is $\bar{z} = 2 + 3i$

Properties

1. $z = 0 \Leftrightarrow \bar{z} = 0$
2. $\bar{z} = \overline{x + iy} = x - iy$
3. $\bar{i} = -i \quad and\ \bar{\bar{i}} = i$
4. $\bar{\bar{z}} = z$
5. $\bar{z} = -z$ (اذا كان العد المعقد خيالي صرف)
6. $\bar{z} = z$ (اذا كان العد المعقد حقيقي)
7. $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$
8. $z + \bar{z} = 2\,Re(z) = 2x$
9. $z - \bar{z} = 2i\,Im(z) = 2iy$
10. $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$
11. $\overline{z_1 . z_2} = \bar{z}_1 . \bar{z}_2$
12. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, \quad z_2 \neq 0$

Exercise to solve

1. Write the following complex numbers in the form $x + iy$ where x and y are real numbers
   - **a)** $(2+3i)^2$
   - **b)** $\frac{1}{i} + \frac{1}{1-i}$
   - **c)** $\frac{7-6i}{2+3i}$
2. Prove that
   - **a)** $\overline{z+3i} = z - 3i$
   - **b)** $\frac{\overline{(2+\iota)^2}}{3-4i} = 1$

Algebraic properties الخصائص الجبرية

ان معظم خصائص الجمع و الضرب بين الاعداد المعقدة تطابق نظيراتها بين الاعداد الحقيقية.

1. **(The Commutative Law) قاعدة الابدال**

   $z_1 + z_2 = z_2 + z_1$ (Addition)

   $z_1 . z_2 = z_2 . z_1$ (Multiplication)

2. **(The Associative Law) قاعدة التجميع**

   $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ (Addition)

   $(z_1 . z_2) . z_2 = z_1 . (z_2 . z_3)$ (Multiplication)

3. **(Distributive Law) قاعدة التوزيع**

   $z_1(z_2 + z_3) = z_1 . z_2 + z_1 . z_3$ توزيع الضرب على الجمع

4. **(Additive Identity) العنصر المحايد الجمعي**

   Since $0 = (0,0) = 0 + i.0$

   $z + 0 = 0 + z = z \ \forall z \in ₵$

5. (Multiplicative Identity) العنـصـــر المحـــايد الضـــربــي

Since $1 = (1,0) = 1 + i.0$ then

$z.1 = 1.z = z \quad \forall z \in ₵$

6. (Additive Inverse) النظيـــر الجمـعـــي

$-z = -x - iy \qquad \forall z \in ₵$

Such that

$z + (-z) = 0$

$-z$ is said to be the additive invers of z

7. (Multiplicative Inverse) النظـــير الجـمعي

$z\,z^{-1} = z^{-1}z = 1 \qquad \forall z \in ₵$

z^{-1} is said to the multiplicative inverse of z

<u>Example:</u>

Find the multiplicative inverse of z

$z = 3i - 2$

<u>Solu.</u> The multiplicative inverse of z is

$$z^{-1} = \frac{1}{z} = \frac{1}{3i-2} = \frac{1}{-2+3i} = \frac{(-2-3i)}{(-2+3i)(-2-3i)}$$

$$\therefore z^{-1} = \frac{-2-3i}{4+9} = \frac{-2-3i}{13} = \frac{-2}{13} - i\,\frac{3}{13}$$

Absolute value الـــقيـمــة المطلـــقــة

القيمة المطلقة للعدد المعقد $z = x + iy$ ويرمز لها بالرمز $|z|$ هي $|z| = \sqrt{x^2 + y^2}$ وهي عدد غير سالب.

Example:

Find the absolute value of the complex number $z = 4 + 3i$

$$|z| = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5$$

Properties:

1. $|z| = \sqrt{z\bar{z}}$
2. $|z| = |\bar{z}|$
3. $|z_1 - z_2| = |z_2 - z_1|$
4. $|z_1 . z_2| = |z_1| . |z_2|$

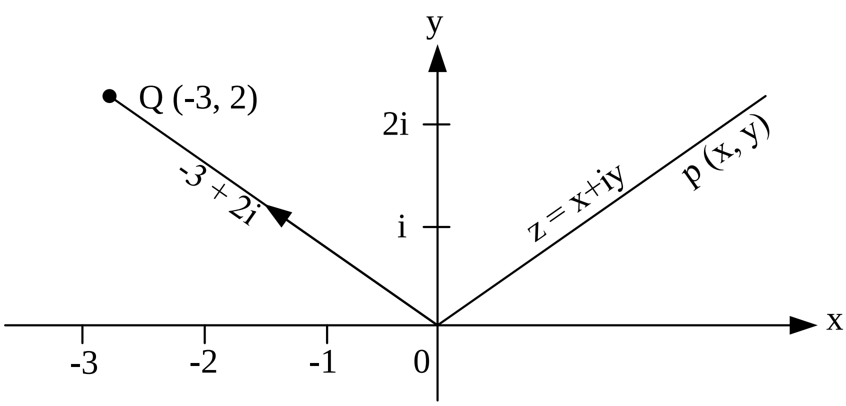
Geometric Representation of a Complex Number

التمثيـــل الهنـدسي للعـــدد المعـقـد

ان كل عدد معقد $z = x + iy$ يقابل نقطة احداثها (x, y) من المستوي xy وبالعكس ان (كل نقطة في المستوي xy تقابل عدداً معقداً) وكل عدد معقد z يمكن تمثيله بمتجه (vector) مبدؤه نقطة الاصل ونهايته النقطة التي تقابل ذلك العدد.

يسمى المحور x بالمحور الحقيقي والمحور y بالمحور الخيالي والمستوي xy بمستوي الاعداد المعقدة او المستوي z.

مـــثـــال: العدد المعقد $z = -3 + 2i$ يقابل النقطة Q(-3, 2) كما في الشكل



مـــلاحـــظـــة

ان المسافة بين نقطتين متمثلتين بالعددين المعقدين z_2, z_1 هي

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

مــــلاحــظـــة

الاعداد المعقدة التي تقابل نقاط واقعة على محيط دائرة مركـزها الاصل ونصـف قطـــرها r (radius) تحقق المعادلة

$$|z| = r$$

والاعداد المعقدة التي تقابل نقاط واقعة على محيط دائرة مركزها النقطة $z_0 = x_0 + iy_0$ ونصف قطرها عدد حقيقي موجب r تحقق المعادلة

$$|z - z_0| = r$$

الأحـــداثيـــات القطـــبـيـــة Polar coordinates

لتكن $\theta, (r \geq 0)$ احداثيات قطبية للنقطة ρ تقابل العدد المعقد غير الصفر $z = x + iy$ لما كان

$$y = r\sin\theta \quad , \quad x = r\cos\theta$$

فإن العدد المعقد z يمكن اعادة كتابته بالصيغة $z = r(\cos\theta + i\sin\theta)$ ان العدد الحقيقي r هو طول المتجهه الذي يمثل z اي ان $r = |z|$ والعدد والحقيقي θ يسمى زاوية العدد المعقد z (argument of z) ويكتب

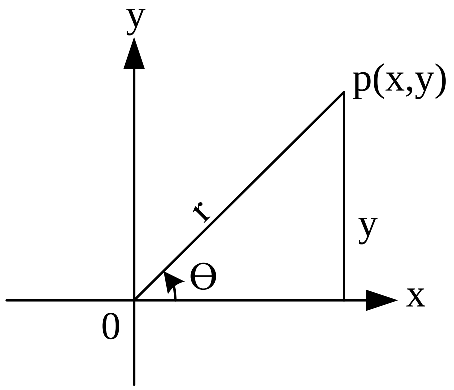
$$\theta = \arg z$$

ان زاوية العدد المعقد z اي الزاوية θ هي الزاوية التي يضعها المتجه z مع محور الموجب باتجاه عكس عقارب الساعة.

وعلى ذلك فان لكل θ عدد غير منته من القيم الحقيقية تختلف عن بعضها بمضاعفات 2 ويمكن ايجاد هذه القيم من المعادلة

$$\tan\theta = \frac{y}{x}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$



Euler's formula صيـــغة اويلـــــر

ان المتطابقة $e^{i\theta} = \cos\theta + i\sin\theta$ تعرف بصيغة اويلر ويمكن كتابة العدد المعقد z بالصيغة

$$z = r\,(\cos\theta + i\sin\theta) = re^{i\theta}$$

ومنها يكون النظير الضربي لـ z هو

$$z^{-1} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$

وحاصل الضرب هو

$$z_1 z_2 = r_1 e^{i\theta_1}.r_2 e^{i\theta_2}$$

$$= r_1 r_2 e^{i(\theta_1+\theta_2)}$$

وناتج القسمه هو

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2}\, e^{i(\theta_1-\theta_2)}$$

Exercise to solve

1. Write the complex numbers in Polar form

(a) $2 + 2\sqrt{3}\,i$ **(b)** $-5 + 5i$ **(c)** $1 + i$ **(d)** $\sqrt{3} - i$

Solu. (b)

$$\boldsymbol{z = -5 + 5i}$$

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{25 + 25} = \sqrt{50} \Rightarrow r = \sqrt{50}$$

$$\theta = \tan^{-1}\left(\frac{x}{y}\right) \Rightarrow \theta = \tan^{-1}\left(\frac{+5}{-5}\right) = \tan^{-1}(-1) = \frac{-\pi}{4}$$

$$z = r(\cos\theta + i\sin\theta)$$

$$z = \sqrt{50}\left(\cos\frac{-\pi}{4}.\,i\sin\frac{-\pi}{4}\right) = \sqrt{50}\;e^{i\frac{\pi}{4}}$$

Solu. (c)

$\boldsymbol{z = 1 + i}$

$z = r = \sqrt{1+1} = \sqrt{2} \quad \Rightarrow r = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4} \quad or \quad \tan(\theta) = i \quad \Rightarrow \theta = \frac{\pi}{4}$

$\therefore z = r(\cos\theta + i\sin\theta)$

$z = re^{i\theta} = \sqrt{2}e^{i\frac{\pi}{4}}$

Solu. (d)

$z = \sqrt{3} - i \quad \Rightarrow |z| = r = \sqrt{3+1} = \sqrt{4} = 2$

$\tan\theta = \frac{y}{x} \Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \quad \Rightarrow \theta = \frac{\pi}{6}$

$\therefore z = re^{i\theta} = 2e^{i\frac{\pi}{6}}$

$or\ z = r(\cos\theta + i\sin\theta) = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

Powers and roots القـــوى والجـــذور

ان القوى الصحيحة لعدد معقد $z = re^{i\theta}$ يمكن ايجاد صيغتها $z^n = r^n e^{in\theta}$

De Moiver's Formula صيغـــة دي مـوفـــيـــري

$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

Ex. Evaluate

(a) $(1+i)^8$

(b) $(1+i)^{-8}$

Solu . (a)

Write $(1 + i)$ in polar form

$|z| = |1+i| = \sqrt{1^2+1^2} = \sqrt{2} = r$

$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4} \quad or \quad \tan\theta = \frac{y}{x} \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

$$\therefore (1+\mathrm{i}) = re^{i\theta} = \sqrt{2}\, e^{i\frac{\pi}{4}}$$

$$\therefore (1+\mathrm{i})^8 = \left(\sqrt{2}e^{i\left(\frac{\pi}{4}\right)}\right)^8 = \left(\sqrt{2}\right)^8 e^{i2\pi} = 2^{\frac{8}{2}} (\cos 2\pi + \ i \sin 2\pi)$$

$$= 2^{\frac{8}{2}} [\cos 2\pi + 0\,]$$

$$= 2^{\frac{8}{2}} (1) == 2^{\frac{8}{2}} = 2^4 = 16\ .$$

<u>Solu . (b)</u>

من الفرع(a) وجدنا : $(1+i)^{-8} = \left(\sqrt{2}\right)^{-8} e^{-2i\pi}$

$= \frac{1}{16}\ (\cos 2\pi - \ i \sin 2\pi) = \frac{1}{16}(1-0) = \frac{1}{16}$

يمكننا استخدام الصيغة $z^n = r^n e^{in\theta}$ في ايجاد جذور الاعداد المعقدة، فمثلاً لايجاد جذور المعادلة $z^n = 1$ نفرض $z = re^{i\theta}$ ولما كان $1 = 1.e^{i\theta}$ فإن

$$\left(re^{i\theta}\right)^n = 1.e^{i\theta}$$

$$r^n e^{in\theta} = 1.e^{i\theta}$$

اذا تساوى عددان معقدان قيمتاهما المطلقتان وان زاويتهما تختلف بمضاعفات 2π أي ان

$$r^n = 1 \quad , \quad n\theta = 0 + 2\mathrm{k}\pi \qquad (k = 0, \quad \pm 1, \quad \pm 2, \ldots)$$

ومنها نجد ان

$$\theta = \frac{2\mathrm{k}\pi}{n} \quad ; \quad r = 1 \qquad (k = 0, \quad \pm 1, \quad \pm 2, \ldots)$$

وبهذا نحصل على الحلول

$$z = e^{i\frac{2\pi k}{n}} = \cos\left(\frac{2\pi k}{n}\right) + i \sin\left(\frac{2\pi k}{n}\right)$$

يمكن تمثيل هذه الجذور بنقاط رؤوس مضلع منتظم عدد اضلاعه n ونصف قطر دائرته الخارجية او مركزه الاصل.

$$w_n = \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right)$$

بموجب مبرهنة دي موفيري تكون الجذور النونية للواحد هي

$$1, w_n, w_n^2, w_n^3, \ldots., w_n^{n-1}$$

لاحظ ان $w_n^n = 1$ ويمكن تعميم هذه الفكرة في حساب الجذور النونية لاي عدد معقد

، $w = p(\cos\rho + i\sin\rho)$، $w \neq 0$ لتصبح بالصيغة

$$\sqrt{w} = \sqrt{p}\left[\cos\left(\frac{\rho + 2k\pi}{n}\right) + i\sin\frac{\rho + 2k\pi}{n}\right]$$

$$(k = 0, 1, 2, \ldots., n-1)$$

حيث $\sqrt[n]{p}$ الجذر النوني الموجب للعدد الحقيقي p الذي يمثل طول كل المتجهات التي تمثل مواقع الجذور النونية

الزاوية $\frac{cp}{n}$ تمثل زاوية العدد المعقد لأحد الجذور.

مــلاحــظـــة

ان زوايا الجذور الاخرى تنتج باضافة مضاعفات$\frac{2\pi}{n}$ الى $\frac{\rho}{n}$ اي لأيجاد جذر عدد معقد، يلزم ايجاد جذر حول متجهه ثم تقسيم زواياه على دليل الجذر واذا كان z_0هو اي من الجذور النونية للعدد المعقد لها فأن مجموع كل الجذور النونية هي

$$\{z_0, z_0 w_n, z_0 w_n^2, z_0 w_n^3, \ldots, z_0 w_n^{n-1}\}$$

حيث

$$w_n = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$$

وان الضرب بــw_n يزيد زاوية العدد المعقد بمقدار $\frac{2\pi}{n}$

مثـــال: جد الجذور التكعيبية الثلاثة للعدد $\left(\sqrt{6} - i\sqrt{2}\right)$

الحـــل : نكتب العدد $\left(\sqrt{6} - i\sqrt{2}\right)$ بالصيغة القطبية فتكون قيمته المطلقة

$$\emptyset = \tan^{-1}\frac{-\sqrt{2}}{\sqrt{6}} = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$

$$r = p = \sqrt{6+2} = 2\sqrt{2}$$

$$\therefore z_0 = \sqrt[3]{2\sqrt{2}\, e^{-\frac{1}{3}i\frac{\pi}{6}}}$$

$$= \sqrt{2}\left[\cos\left(\frac{\pi}{18}\right) - i\sin\left(\frac{\pi}{18}\right)\right]$$

$$= \sqrt{2}[\cos 10^\circ - i\sin 10^\circ] = \sqrt{2}[\cos(-10) + i\sin(-10)]$$

ان الجذرين الاخرين ينتجان باضافة الى الزاوية (10-) ما يعادل $\frac{1}{3}$ الدورة الواحدة اي$\frac{360}{3} = 120$ فيكون

$$z_0 w_3 = \sqrt{2}[\cos(-10^\circ) - i\sin(-10^\circ)]\,[\cos(120^\circ) + i\sin(120^\circ)]$$

$$= \sqrt{2}[\cos(110^\circ) + i\sin(110^\circ)] \quad = \sqrt{2}\, e^{i\left(\frac{110\pi}{180}\right)}$$

$$z_0 w_3^2 = \sqrt{2}[\cos(-10^\circ) + i\sin(-10^\circ)][\cos(120^\circ) + i\sin(120^\circ)]^2$$

$$= \sqrt{2}\,[\cos(230) + i\sin(230)]$$

$$= \sqrt{2}\, e^{i\left(\frac{230}{180}\right)\pi}$$

<u>تمارين</u>

1. جد قيمة واحدة الزاوية العدد المعقد z عندما

$$(\boldsymbol{a})\;\; z = \frac{-2}{1 + i\sqrt{3}} \qquad\qquad (\boldsymbol{b})\; z = \frac{i}{-2 - 2i}$$

2. جد كل الجذور $(-i)^{1/3}$

3. جد قيم كل الجذور

$$(\boldsymbol{a})\;\; \sqrt{-i} \qquad\qquad (\boldsymbol{b})\; \sqrt{1 + i}$$

Exercise to solve

<u>Q.1 /</u> Solve the following equation, for the real numbers x and y

(a) $(3 + 4i)^2 - 2(x - iy) = x + iy$

(b) $\left(\frac{1+i}{1+i}\right)^2 + \frac{1}{x+iy} = 1 + i$

(c)$(3 - 2i)(x + iy) = 2(x - 2iy) + 2i - 1$

<u>Q.2 /</u> How many the following complex numbers be obtained from $z = x + iy$ geometically? Sketch

(a) $-\bar{z}$ **(b)** $(\overline{-z})$

(c)$\bar{z}$ **(d)** $1/z$

Q.3 / Find the result of:

(a) $\frac{\mathbf{3-i}}{\mathbf{4+5i}}$ **(b)** $\frac{(1+i)(2-i)}{(1-i)(3+i)}$

Q.4 / Show that the distance between the two points z_1 and $z_2$in an a rgand diagram is equal to $|z_1 - z_2|$

Q. 5 / In the following problems, graph the points $z = x + iy$ that satisfy the given conditions

(a) $|z| = 2$ **(b)** $|z| < 2$

(c) $|z| > 2$ **(d)** $|z + 1| = 1$

Q.6 / Express the answer to the following problems in the form $re^{i\theta}$ with $r \geq 0$ and $-\pi \leq \theta \leq \pi$. Sketch

(a) $\left(1 + \sqrt{-3}\right)^2$ **(b)** $\frac{1+i}{1-i}$

Q.7 / Use De Moivre's theorem to express $\cos 4\theta$ and $\sin 4\theta$ as polynomials in $\cos \theta$ and $\sin \theta$

Q.8 / Find the results of the following:

(a) $\sqrt{i}$ **(b)** $\sqrt{-1 - \sqrt{3}i}$

Q.9 / Find the square roots at $(1 + i)^3$

Q.10 / Find the cube roots of the following

(a) $unity$ **(b)** -8

(c)$-5 + 2i$ **(d)** $1 + \sqrt{3}i$

Q.11 / Find the roots of the following equation

$$2x^2 - 4x + 4 = 0$$

Q.12 / Find the four roots of the following equations

(a) $x^4 + 1 = 0$

(b) $z^4 - 2z^2 + 4 = 0$

Q.13 / Find

(a) $|i(2-i)^3|$ **(b)** $\left|\frac{2+3i}{3+4i}\right|$

Q.14 / Write the following complex numbers in form $x+iy$

(a) $3e^{\frac{\pi}{3}i}$ **(b)** $2e^{-\frac{\pi}{4}i}$

(c)$e^{\frac{\pi}{3}}\ e^{-\frac{\pi}{6}i}$ **(d)** $-5\ e^{-\frac{7}{5}\pi i}$

اســئلـــة نصــف الـــســنة اعــوام ســابــقــة

Al – Mansour University College Computer Communications Eng. Dept Year: 1^{st} Examiner: Dr. Emad A	Subject : Mathematics I Date: 2010-1-13 Time: 2 hours

Q.1 (a) solve the system by invests matrix method:

$x_1 + 2x_3 = 6$

$-3x_1 + 4x_2 + 6x_3 = 30$

$-x_1 - 2x_2 + 3x_3 = 8$

(b) Find the asymptotes of the function $y = \tanh\left(\ln\sqrt{x}\right)$

Q.2 (a) if $y = (\sinh^{-1})^{e^x}.\ find\ \frac{de^{\sin^{-1} y}}{d\ \ln x}$

(b) Write the following complex number in polar form:

(1) $\frac{1+\sqrt{3}i}{(1-i)^2}$ **(2)** $\frac{1}{1+i}$

Q.3 (a) show that $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = e^{ab}$**, where a and b are constants**

(b) Solve for y: if

$$3^{\log_3 7} + 2^{\log_2 5} = 5^{\log_5 y}$$

Q.4 (a) find the domain and the range of the following functions:

(1) $y = f(x) = \ln(1-x^2)$, **(2)** $y = f(x) = \sqrt{2-\sqrt{x}}$

(b) Prove that:

If $|z| = 1$, then $Re\left(\frac{\bar{z}}{1-\bar{z}}\right) = -\frac{1}{2}$

مــلاحظة: تـوزع الـــــدرجات بالتــساوي على الاســئلة

Al – Mansour University College **Computer Communications Eng. Dept** **Year: First** **Examiner: Dr. Emad A**	**Subject : Mathematics I** **Date: 2011-2-1** **Time: 2 hours**

مــلاحظة: الاجــابــة على جميــع الاســئلــة

<u>Q.1</u> Solve the system by (inverse matrix method):

$2x_1 + \ln(x_2) - x_3 = 4$

$x_1 + \ln(\frac{1}{x_2^2}) + x_3 = -10$

$-3x_1 - 2x_3 = 9$

<u>Q.2</u> Find $\frac{dy}{dx}$ $\boldsymbol{if}$ $\boldsymbol{y = \ln\left[\log_7\left(\frac{\sin h^{-1}x \ \cos h^{-1}x}{e^x \quad 3^x}\right)\right]}$

<u>Q.3</u> (a) Evaluate: $\boldsymbol{\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{3x}}$

(b) Write the following complex number in polar form: $\frac{1+\sqrt{3}i}{(1-i)^2}$

<u>Q.4</u> Find the domain of the functions:

(1) $y = \ln\left(\frac{x^2-1}{2x}\right)$, **(2)** $y = \cos^{-1}(\sin x)$, **(3)** $y = \sqrt{2-\sqrt{x}}$

<u>Q.5</u> (a) if $\boldsymbol{y = (\cos^{-1} h^{-1})^{e^x}. find \ \frac{de^{\sec^{-1} y}}{d \ln x}}$

(b) Solve the equation:

$$2^{\log_2(x^2)} + 4e^{\ln(x)} - 3 \,.\, 10^{\log_{10}(1)}$$

مــع تــمنيــاتي لــكــم بــالنــجــاح