Al-Mansour University College

قسم الهندسة المدنية
المرحلة الثانية

Civil Eng. Dept.
2nd. Stage

Engineering Mathematics

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Lec.1

الرياضيات

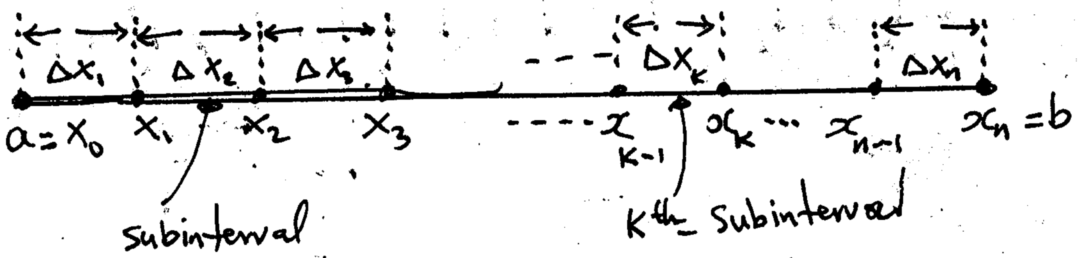
The Definite Integral

A partition of the closed interval $[a,b]$ is a collection of points,

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

that divides $[a,b]$ into $\underline{n}$ subinterval of lengths

$$\Delta x_1 = x_1 - x_0 \;, \quad \Delta x_2 = x_2 - x_1 \;, \quad \ldots \;, \quad \Delta x_n = x_n - x_{n-1}$$



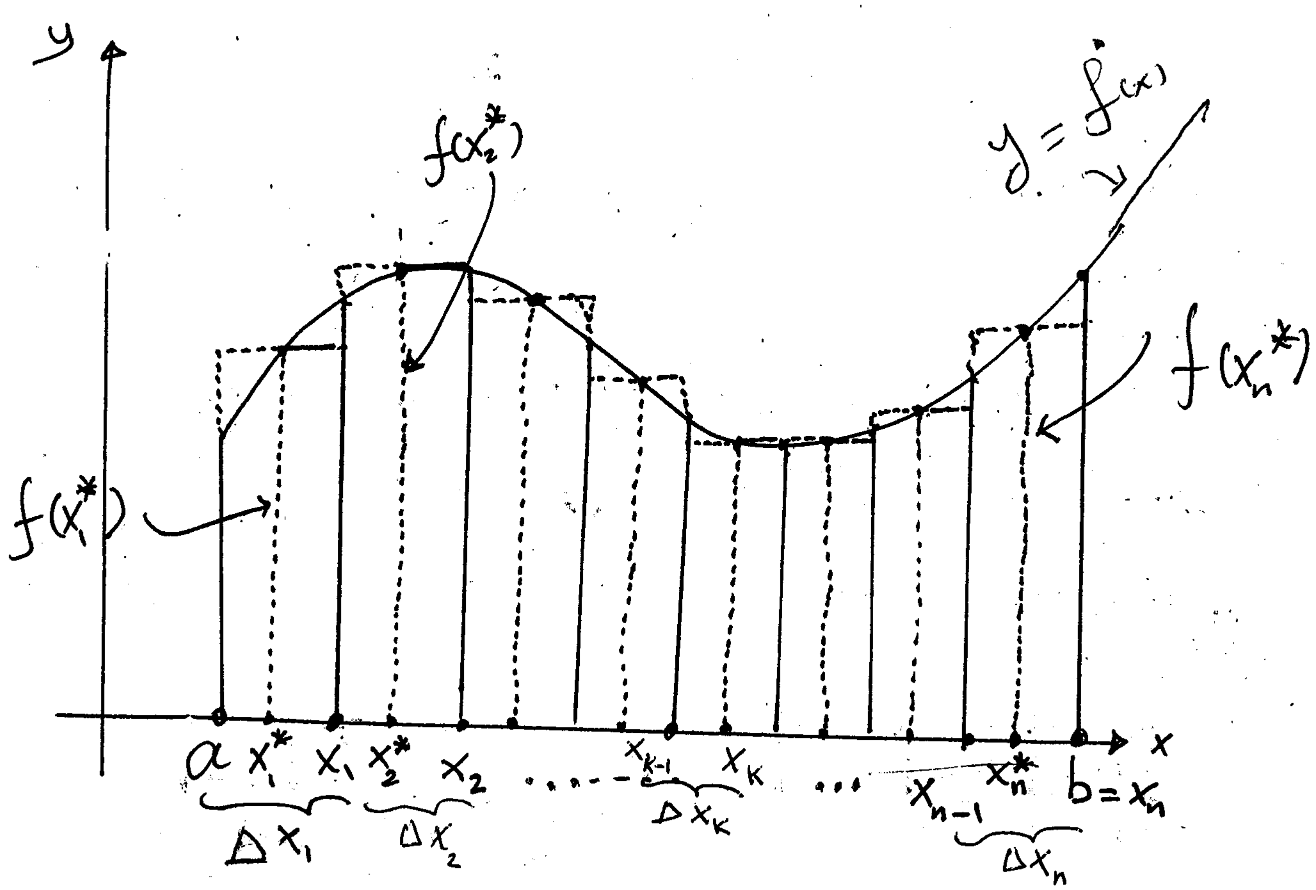
The partition is said to be regular provided the subintervals <u>all</u> have the <u>same</u> length

$$\Delta x_k = \Delta x = \frac{b-a}{n},$$

<u>Def.</u> If the function f is continuous on $[a,b]$, then the <u>net signed area</u> A between $y = f(x)$ and the interval $[a,b]$ is defined by

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*) \, \Delta x \leftarrow \text{regular}$$

where x_k^* is an arbitrary selected point in the equal subinterval widths.



If we have unequal subinterval widths, then the net signed area A is

$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*) \, \Delta x_k$$

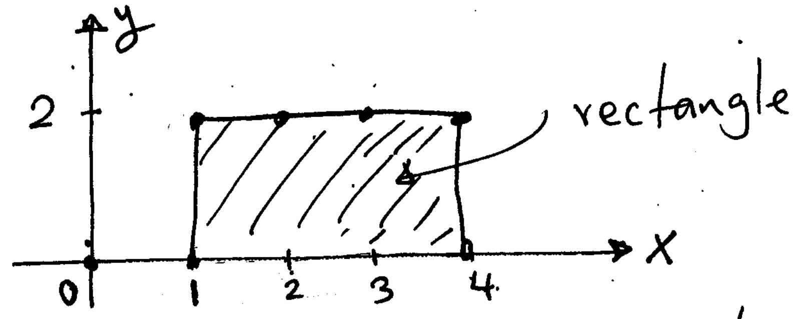
<u>Def</u>. A function f is said to be <u>integrable</u> on a finite closed interval $[a,b]$ if the limit

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*) \, \Delta x_k$$

exists and does not depend on the choice of partitions or the choice of the points x_k^* in the

Sol. $y = f(x) = 2$ and the interval is $[1, 4]$

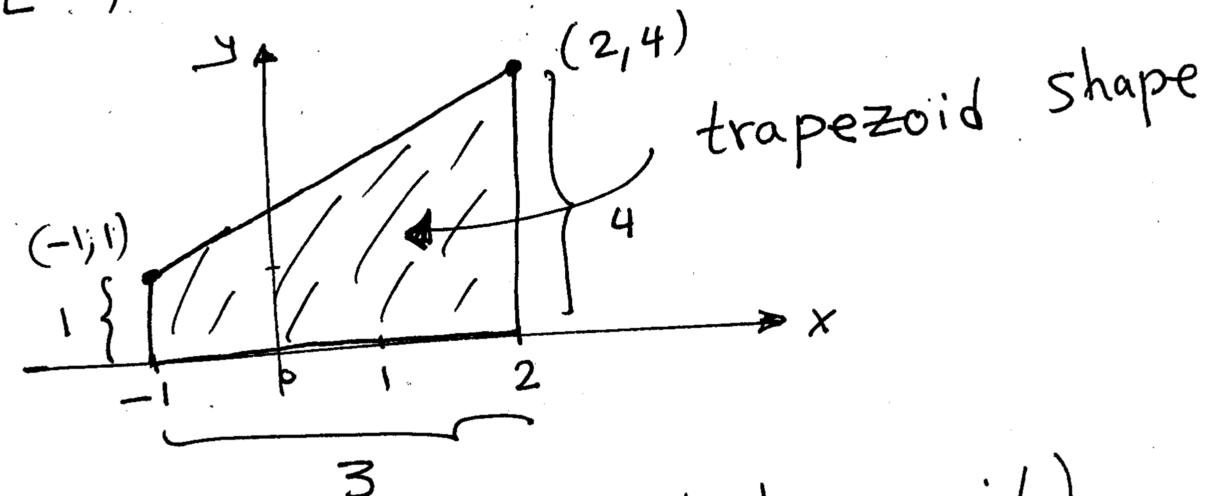
$\underline{4-1=3}$



$$\Rightarrow \int_1^4 2\,dx = (\text{area of rectangle}) = 2(3) = 6$$

(2) $\displaystyle\int_{-1}^{2} (x+2)\,dx$

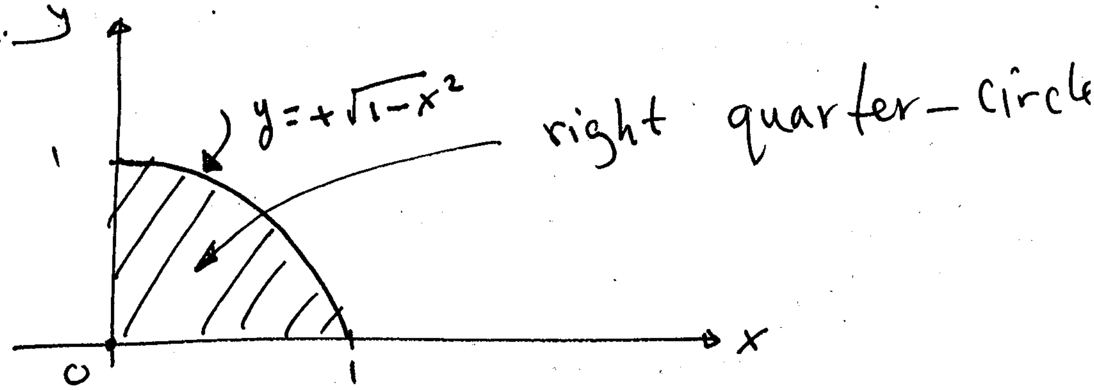
Sol. $y = f(x) = x+2$, and the interval is $[-1, 2]$



$$\Rightarrow \int_{-1}^{2} (x+2)\,dx = (\text{area of trapezoid})$$
$$= \left(\frac{1+4}{2}\right)(3) = \frac{15}{2}.$$

(3) $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$.

Sol. The graph of $y = +\sqrt{1-x^2}$ is the upper semicircle of raduis 1, centered at the origin, Since the interval $[0,1]$, so the region is the <u>right quarter-circle</u>,

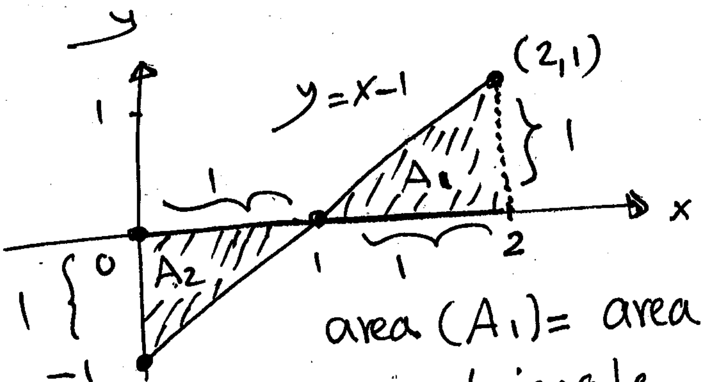


$$\Rightarrow \int_0^1 \sqrt{1-x^2}\, dx = (\text{area of quarter-circle}$$

$$= \frac{1}{4}(1)^2 \pi = \frac{\pi}{4}.$$

(4) $\displaystyle\int_0^2 (x-1)\, dx$

Sol. $y = f(x) = x - 1$



$\Rightarrow$ The net signed area is $A_1 - A_2 = \frac{1}{2} - \frac{1}{2} = 0$

$$\Rightarrow \int_0^2 (x-1)\, dx = 0, \quad \text{and}$$

area (A_1) = area of triangle

$= \frac{1}{2}(1)(1) = \frac{1}{2}$

area (A_2) = area of triangle

$= \frac{1}{2}(1)(1) = \frac{1}{2}$

(7) If $f(x) \geq 0$ for all x in $[a,b]$, then

$$\int_a^b f(x)\,dx \geq 0,$$

and if $f(x) \leq 0$ for all x in $[a,b]$, then

$$\int_a^b f(x)\,dx \leq 0$$

(8) If $f(x) \geq g(x)$ for all x in $[a,b]$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

<u>Theorem</u> (<u>The Fundamental Theorem of Calculus</u>)

If f is continuous on $[a,b]$ and F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

<u>Ex</u>. $\int_1^2 x\,dx = \frac{1}{2}x^2\Big]_1^2 = \frac{1}{2}(2)^2 - \frac{1}{2}(1)^2$
$= 2 - \frac{1}{2} = \frac{3}{2}$.

<u>Ex</u>. $\int_0^3 (9-x^2)\,dx = 9x - \frac{1}{3}x^3\Big]_0^3 = 9(3) - \frac{1}{3}(3)^3$
$= 27 - 9 = 18$.

<u>Ex</u>. $\int_0^{\pi} \cos x\,dx = \sin x\Big]_0^{\pi} = \sin\pi - \sin(0) = 0$.

Ex. $\displaystyle\int_0^{\pi/3} \sec^2 x \, dx = \tan x \Big]_0^{\pi/3} = \tan\frac{\pi}{3} - \tan 0$
$= \sqrt{3} - 0 = \sqrt{3}$

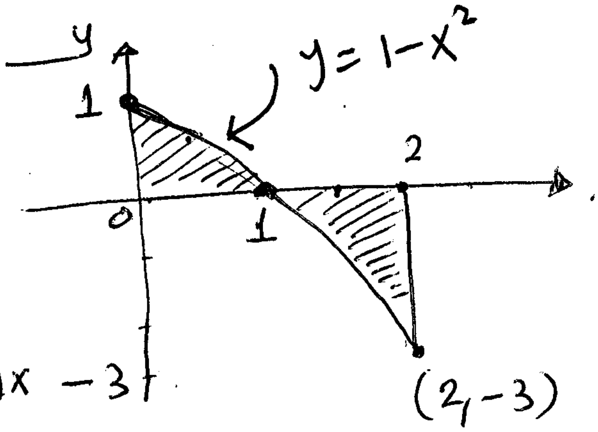
Ex. $\displaystyle\int_{-\frac{\pi}{4}}^{\pi/4} \sec x \tan x \, dx = \sec x \Big]_{-\pi/4}^{\pi/4} = \sec\frac{\pi}{4} - \sec\left(\frac{-\pi}{4}\right)$
$= \sqrt{2} - \sqrt{2} = 0$

Total Area

If f is continuous function on the interval $[a,b]$, then we define the total area between the curve $y = f(x)$ and the interval $[a,b]$ to be

$$\text{Total Area} = \int_a^b |f(x)| \, dx .$$

Ex. Find the total area between the curve $y = 1 - x^2$ and x-axis over the interval $[0,2]$.



sol. Let A be the total area, then

$$A = \int_0^2 |1 - x^2| \, dx$$

$$= \int_0^1 (1 - x^2) \, dx + \int_1^2 -(1 - x^2) \, dx$$

$$= \int_0^1 (1 - x^2) \, dx + \int_1^2 (x^2 - 1) \, dx$$

$y = 0 \Rightarrow 1 - x^2 = 0$
$\Rightarrow x^2 = 1 \Rightarrow x =$ $\boxed{x = +1}$

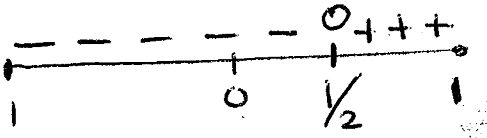
يهمل $x = -1$ لأنه خارج الفترة المحددة بالسؤال.

$$A = \left(x - \tfrac{1}{3}x^3\right)\Big|_0^1 + \left(\tfrac{1}{3}x^3 - x\right)\Big|_1^2$$

$$= \left(1 - \tfrac{1}{3}\right) + \left(\tfrac{1}{3}(2)^3 - 2\right) - \left(\tfrac{1}{3} - 1\right)$$

$$= \frac{2}{3} + \underbrace{\frac{8}{3} - 2} + \frac{2}{3} = \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = \frac{6}{3} = 2$$

Ex. $\displaystyle\int_{-1}^{1} |2x-1|\,dx$, $2x-1=0 \Rightarrow x = \frac{1}{2}$



$$= \int_{-1}^{1/2} -(2x-1)\,dx + \int_{1/2}^{1} (2x-1)\,dx$$

$$= \left(x - \cancel{2}\frac{x^2}{\cancel{2}}\right)\Big]_{-1}^{1/2} + \left(\cancel{2}\frac{x^2}{\cancel{2}} - x\right)\Big]_{1/2}^{1}$$

$$= \left(\tfrac{1}{2} - \tfrac{1}{4}\right) - \underbrace{(-1-1)} + \underbrace{\cancel{(1-1)}}_{\text{zero}} - \left(\tfrac{1}{4} - \tfrac{1}{2}\right)$$

$$= \frac{1}{4} + 2 + \frac{1}{4} = 2 + \frac{1}{2} = 5/2$$

Th. $\displaystyle\int_a^b f(x)\,dx = \int f(x)\,dx \Big]_a^b$

$$= F(x)\Big]_a^b$$

$$= F(b) - F(a)$$

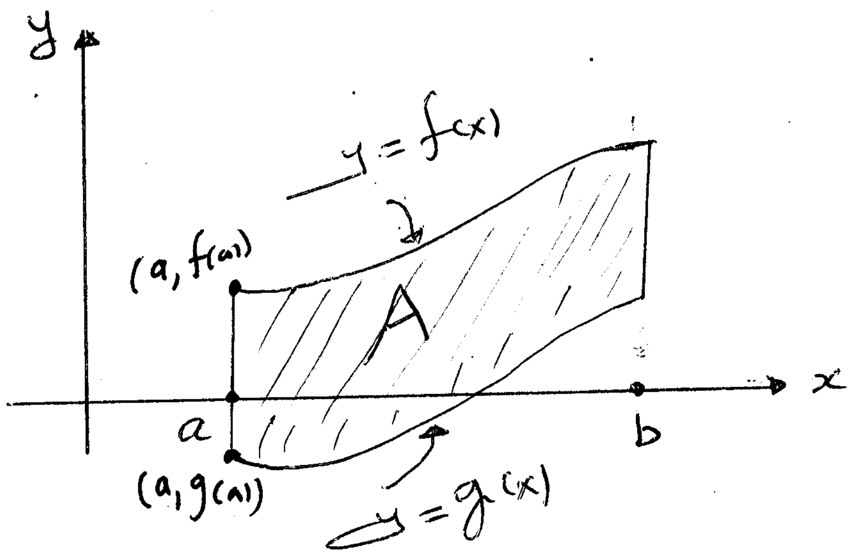
Applications of the Definite Integral

1. Area Between Two Curves

Suppose that f and g are continuous functions on an interval $[a, b]$ and

$$f(x) \geq g(x) \quad \text{for} \quad a \leq x \leq b$$

Find the area A of the region bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by lines $x = a$ and $x = b$.



To solve this problem we divide the interval $[a, b]$ into n subintervals, which has the effect of subdividing the region into n strips

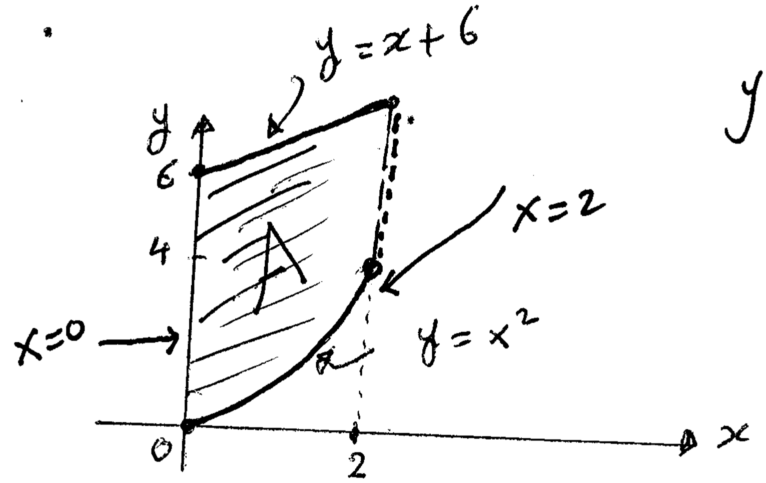
Ex.(1) Find the area of the region bounded above by $y = x + 6$, bounded below by $y = x^2$ and bounded on the sides by the lines $x = 0$ and $x = 2$.

$x = a$

$x = b$

$y = mx + b$

Sol.



$\int x^n dx = \frac{x^{n+1}}{n+1} + \cancel{C}$

$$A = \int_{0}^{2} (x + 6 - x^2)\, dx = \frac{1}{2}x^2 + 6x - \frac{1}{3}x^3 \Big]_0^2$$

b, a

$$= \frac{1}{2}(4) + 6(2) - \frac{8}{3} \cancel{-f(0)}$$

$$= 14 - \frac{8}{3} = \frac{42 - 8}{3} = \frac{34}{3}$$

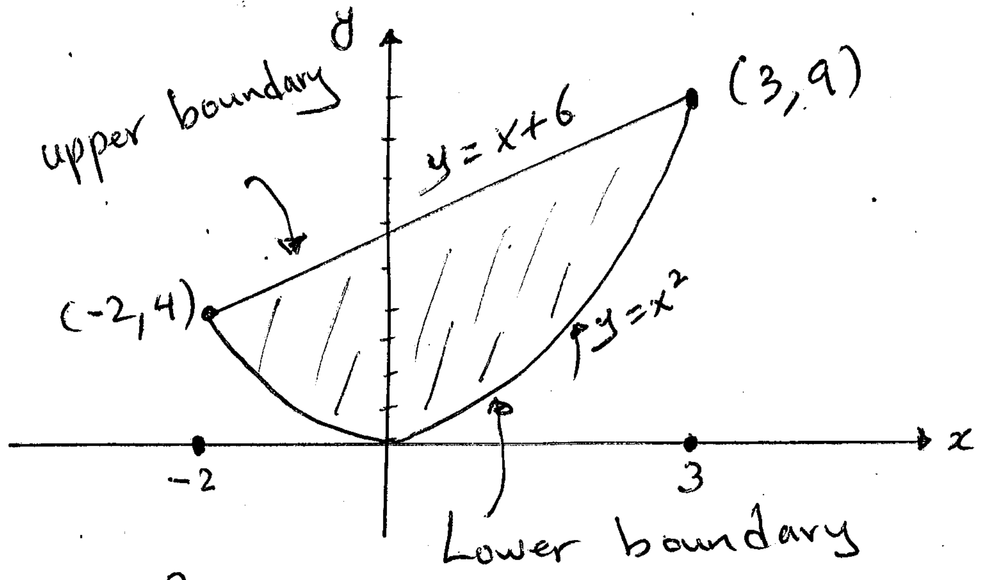
Ex.(2) Find the area of the region that is bounded between the curves $y = x^2$ and $y = x + 6$.

Sol. Solve $\underline{x^2 = x + 6}$ to find the endpoints of the region:

$$x^2 - x - 6 = 0$$

$$(x - 3)(x + 2) = 0 \Rightarrow x = -2,\ x = 3$$

At these points we have $y = 4$ and $y = 9$ respectively,



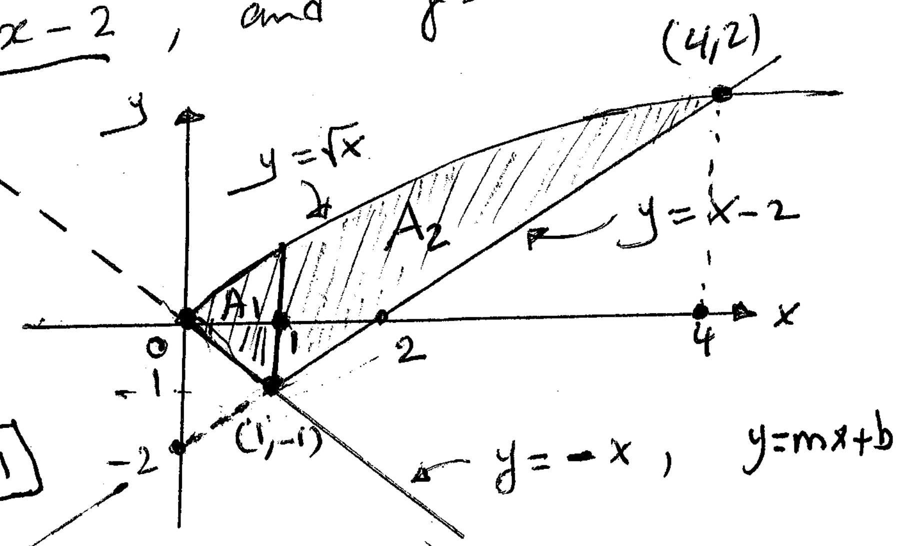
$$\Rightarrow \quad A = \int_{-2}^{3} (x+6-x^2)\,dx$$

$$= \frac{1}{2}x^2 + 6x - \frac{1}{3}x^3 \Big]_{-2}^{3}$$

$$= \underbrace{\frac{1}{2}(3)^2 + 6(3) - \frac{1}{3}(3)^3} - \underbrace{\left(\frac{1}{2}(-2)^2 + 6(-2) - \frac{1}{3}(-2)^3\right)}$$

$$= \frac{27}{2} - \left(\frac{-22}{3}\right) = \frac{125}{6}.$$

Ex.(3) Find the area of the region enclosed by $y=\sqrt{x}$, $y = x-2$, and $y=-x$.



Sol.
To find the points of intersection between $y=\sqrt{x}$ and $y = x-2$,

Solve $\sqrt{x} = x-2 \Rightarrow \boxed{1=-1}$

$x = x^2 - 4x + 4$

or $x^2 - 5x + 4 = 0 \Rightarrow (x-4)(x-1) = 0 \Rightarrow$ $x=4$ and $y=2$

$x = 1$, neglagte

and between $y = x-2$ and $y = -x$, solve

$$x - 2 = -x$$

$$2x = 2 \implies x = 1 \quad \text{and} \quad y = -1.$$

It is clear that, for $0 \le x \le 1$, the upper boundary is $y = \sqrt{x}$ and the lower boundary is $y = -x$ and for $1 \le x \le 4$, the upper boundary is $y = \sqrt{x}$ and the lower boundary is $y = x - 2$. Because of this change in the formula for the lower boundary it is necessary to divide the region into two parts, and find the area of each part separately.

Thus

$$A_1 = \int_0^1 (\sqrt{x} - (-x))\,dx = \int_0^1 (\sqrt{x} + x)\,dx$$

$$= \frac{2}{3} x^{3/2} + \frac{1}{2} x^2 \Big]_0^1 = \frac{2}{3} + \frac{1}{2} = \frac{7}{6}.$$

and

$$A_2 = \int_1^4 (\sqrt{x} - (x-2))\,dx = \int_1^4 (\sqrt{x} - x + 2)\,dx$$

$$= \frac{2}{3} x^{3/2} - \frac{1}{2} x^2 + 2x \Big]_1^4 = \frac{2}{3}(4)^{3/2} - \frac{1}{2}(4)^2 + 2(4) - \left[\frac{2}{3} - \frac{1}{2} + 2\right]$$

$$= \left(\frac{16}{3} \cancel{-8} \cancel{+8}\right) - \left(\frac{2}{3} + \frac{3}{2}\right) = \frac{16}{3} - \frac{13}{6} = \frac{19}{6}$$

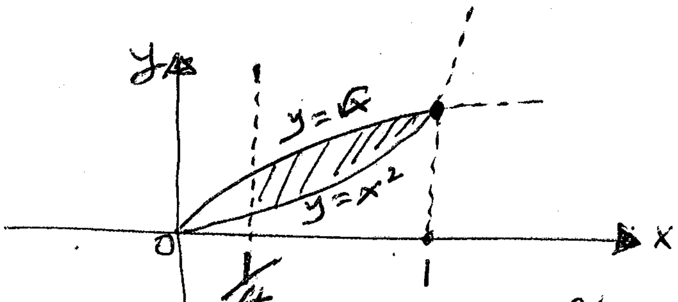
Thus, the area of the entire region is

$$A = A_1 + A_2$$
$$= \frac{7}{6} + \frac{26}{6} = \frac{33}{6}.$$

Ex (4) sketch the region enclosed by the curves and find its area.

(1) $y = x^2$, $y = \sqrt{x}$, $x = \frac{1}{4}$, $x = 1$

sol.

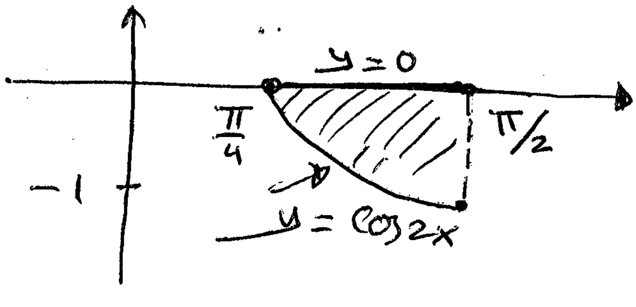


$$A = \int_{1/4}^{1} [\sqrt{x} - x^2]\, dx = \frac{2}{3} x^{3/2} - \frac{1}{3} x^3 \Big]_{1/4}^{1}$$

$$= \left[\frac{2}{3} - \frac{1}{3}\right] - \left[\frac{2}{3}\left(\frac{1}{4}\right)^{3/2} - \frac{1}{3}\left(\frac{1}{4}\right)^3\right]$$

$$= \frac{1}{3} - \frac{2}{24} + \frac{1}{3(64)} = \frac{64 - 16 + 1}{192} = \frac{49}{192}$$

(2) $y = \cos 2x$, $y = 0$, $x = \frac{\pi}{4}$, $x = \frac{\pi}{2}$

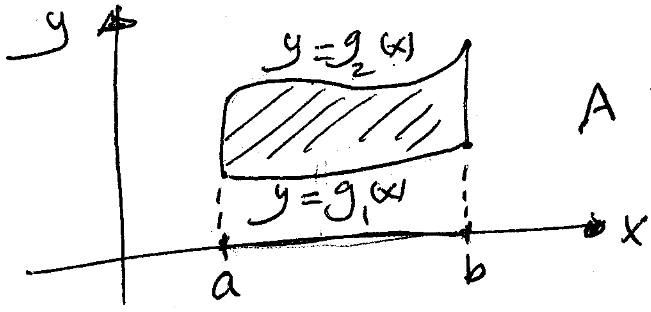


Double Integrals Over Nonrectangular Regions

Our study of double integrals over nonrectangular regions will be limited to two types of regions, which we will call type I and type II; they are defined as follows:

Definition

(a) A type I region is bounded on the left and right by vertical lines $x = a$ and $x = b$ and is bounded below and above by continuous curves $y = g_1(x)$ and $y = g_2(x)$, where $g_1(x) \le g_2(x)$ for $a \le x \le b$.



A type I region

(b) A type II region is bounded below and above by horizontal lines $y = c$ and $y = d$ and is bounded on the left and right by continuous curves $x = h_1(y)$ and $x = h_2(y)$, satisfing $h_1(y) \le h_2(y)$ for $c \le y \le d$.

$$A = \int_{\pi/4}^{\pi/2} [0 - \cos 2x]\, dx = -\int_{\pi/4}^{\pi/2} \cos 2x \cdot dx$$

$$= -\frac{1}{2} \sin 2x \Big]_{\pi/4}^{\pi/2} = \underbrace{\cancel{\frac{-1}{2} \sin \pi}}_{\text{zero}} + \frac{1}{2} \sin \frac{\pi}{2} = \frac{1}{2}$$

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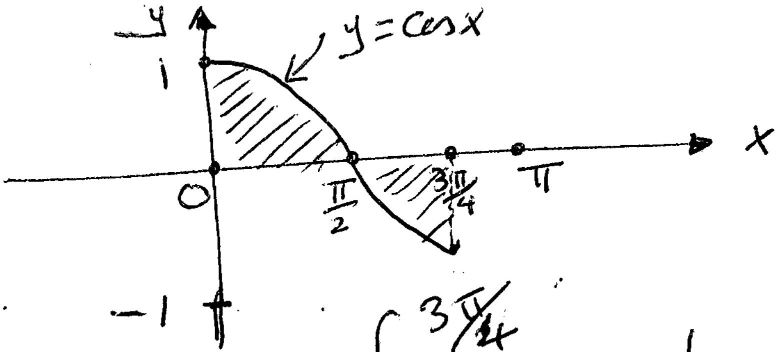
(3) $y = \cos x$, $y = 0$, $x = 0$, $x = \frac{3\pi}{4}$

sol.



$$A = \int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{3\pi/4} -\cos x\, dx$$

$$= \sin x \Big]_0^{\pi/2} - \sin x \Big]_{\pi/2}^{3\pi/4}$$

$$= \sin \frac{\pi}{2} - \sin 0 - \left(\sin \frac{3\pi}{4} - \sin \pi/2 \right)$$

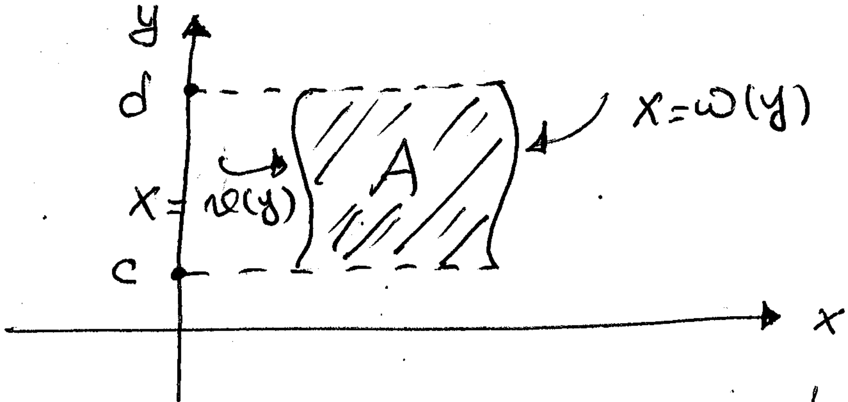
$$= 1 - 0 - \frac{1}{\sqrt{2}} + 1 = 2 - \frac{1}{\sqrt{2}} = \frac{2\sqrt{2} - 1}{2}$$

(2) Second Area Problem

Suppose that w and v are continuous functions of y on an interval $[c, d]$ and that:

$$w(y) \geq v(y) \quad \text{for} \quad c \leq y \leq d.$$

Find the area A of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, and above by the line $y = d$ and below by the line $y = c$



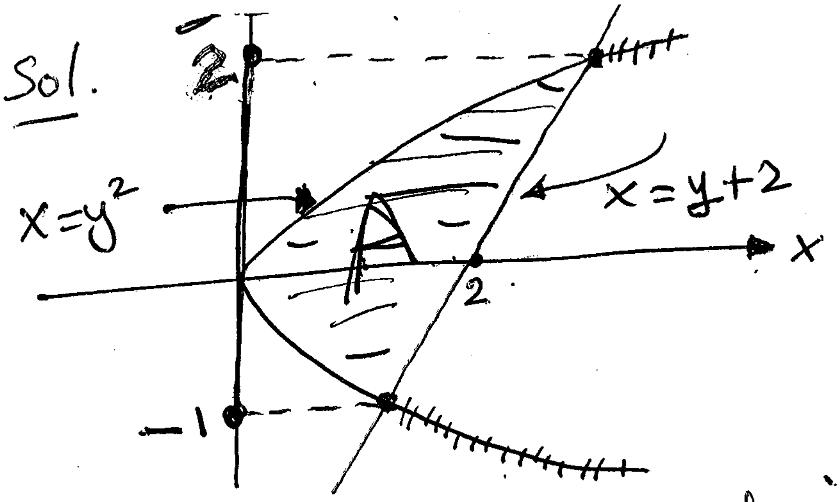
By the same procedure in the derivation of formula (1), but with the roles of x and y reversed, we obtain

$$A = \int_c^d [w(y) - v(y)]\, dy.$$

Ex.(1) Find the area of the region enclosed by $x = y^2$ and $y = x - 2$, integrating with respect to y

$$x = y + 2$$

Sol.



$y = x-2$
with slope $= +1$

$y = mx+b$

To find the points of intersection, solve

$$y^2 = y+2 \Rightarrow y^2 - y - 2 = 0$$

$$\Rightarrow (y-2)(y+1) = 0$$

$$\Rightarrow y = 2, \quad y = -1.$$

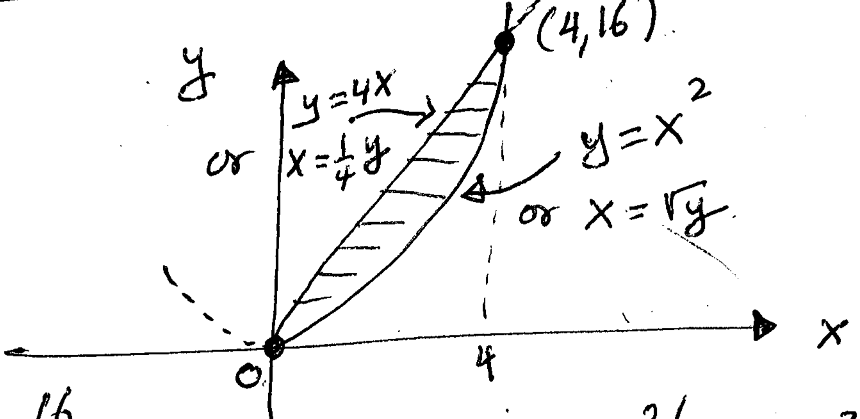
$$\therefore A = \int_{-1}^{2} \left[y+2-y^2\right] dy = \frac{1}{2}y^2 + 2y - \frac{1}{3}y^3 \Big]_{-1}^{2}$$

$$\int y^n dy = \frac{y^{n+1}}{n+1} \quad n \neq -1$$

$$= \frac{9}{2}.$$

(2) $y = x^2$ and $y = 4x$,

Sol.

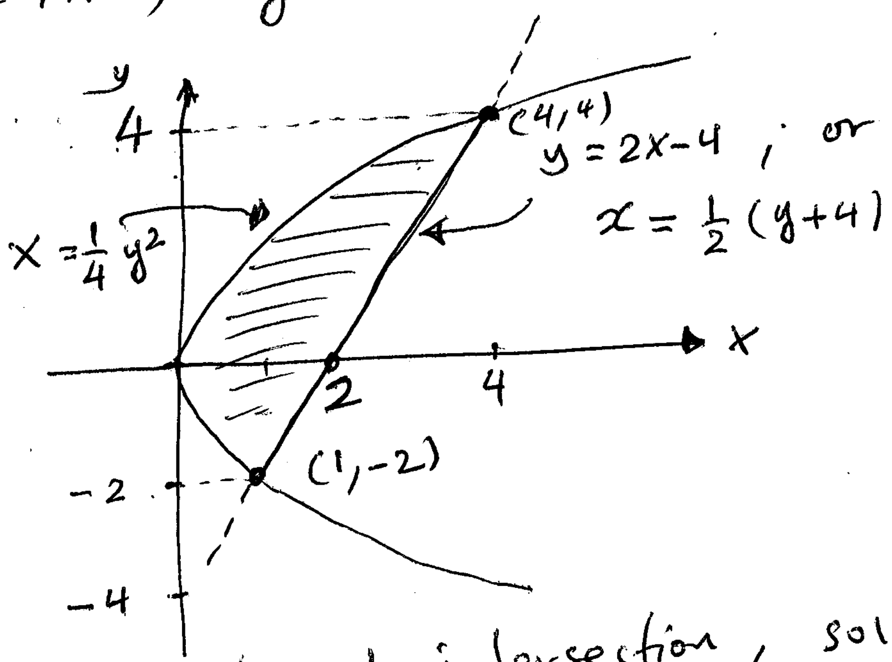


Solve
$x^2 = 4x$
$x^2 - 4x = 0$
$x(x-4) = 0$
$\Rightarrow x=0, x=4$

$$\Rightarrow A = \int_0^{16} \left(\sqrt{y} - \frac{1}{4}y\right) dy = \frac{2}{3}y^{3/2} - \frac{1}{8}y^2 \Big]_0^{16} = \frac{32}{3}.$$

(3) $y^2 = 4x \ , \quad y = 2x-4$

Sol.



To find the points of intersection, solve

$$\frac{1}{4}y^2 = \frac{1}{2}(y+4)$$

$$y^2 = 2y+8 \Rightarrow y^2 - 2y - 8 = 0$$

$$\Rightarrow (y-4)(y+2) = 0 \Rightarrow y = -2 \ , \ y = 4$$

$$A = \int_{-2}^{4} \left[\frac{1}{2}(y+4) - \frac{1}{4}y^2\right] dy$$

$$= \frac{1}{4}y^2 + 2y - \frac{1}{12}y^3 \Big]_{-2}^{4} = \frac{1}{4}(4)^2 + 2(4) - \frac{1}{12}(4)^3 - \left(\frac{1}{4}(-2)^2 + 2(-2) - \frac{1}{12}(-2)^3\right)$$

$$= \left(12 - \frac{64}{12}\right) - \left(1 - 4 + \frac{8}{12}\right)$$

$$= 12 - \frac{16}{3} + 3 - \frac{2}{3} = \frac{27}{3} = 9.$$

If we choose an arbitrary point (x_k, y_k) in the k^{th} subrectangle, then the volume of a rectangular parallelepiped with base area ΔA_k and heigh $f(x_k, y_k)$, is

$$V_k = f(x_k, y_k)\, \Delta A_k$$

If we follow this procedure for all the sub-rectangles and add the volumes, we get an approximation to the total Volume of S

$$V \approx \sum_{k=1}^{n} V_k = \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k \quad \cdots (1)$$

If we refine the mesh width to make $(\Delta x_k, \Delta y_k) \to (0$ the sum in (1) approach a limit called the double integral of f over R, i.e.,

$$\iint_R f(x,y)\, dA = \lim_{\Delta A \to 0} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k = V$$

if the limit exists.

Properties of the Double Integral

(1) $\iint_R k f(x,y)\, dA = k \iint_R f(x,y)\, dA$, (any number k)

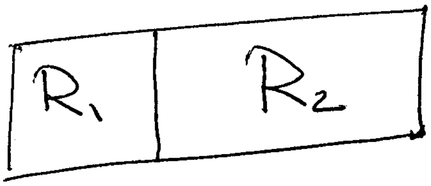
(2) $\iint_R [f(x,y) + g(x,y)]\, dA = \iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA$

(3) $\iint_R [f(x,y) - g(x,y)]\, dA = \iint_R f(x,y)\, dA - \iint_R g(x,y)\, dA$

(4) $\iint_R f(x,y)\, dA \geq 0$, if $f(x,y) \geq 0$ on R

(5) $\iint_R f(x,y)\, dA \geq \iint_R g(x,y)\, dA$ if $f(x,y) \geq g(x,y)$ on R

(6) $\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA$



Evaluating Double Integrals

It is known that the partial derivatives of a function $f(x,y)$ are calculated by holding one of the variables fixed and differentiating with respect to the other variable. Let us consider the reverse of this process, partial integration. The symboles

$$\int_a^b f(x,y)\, dx \quad \text{and} \quad \int_c^d f(x,y)\, dy$$

denote partial definite integrals; the first integral, called the partial definite integral with respect to x, is evaluated by holding y fixed and integrating with respect to x, and the second integral, called the partial definite integral with respect to y, is evaluated by holding x fixed and integrating with respect to y.

Example Evaluate the following

$$\# \int_0^1 x\,y^2\,dx = y^2 \int_0^1 x\,dx = \frac{y^2x^2}{2}\Big]_0^1 = \frac{1}{2}y^2 \text{ (fun. of } y\text{)}$$

$$\# \int_0^1 x\,y^2\,dy = x \int_0^1 y^2\,dy = \frac{x\,y^3}{3}\Big]_0^1 = \frac{x}{3} \text{ (fun. of } x\text{)}$$

The first integral can be integrated with respect to x, and the second integral can be integrated with respect to y. This two-stage integration process is called iterated (or repeated) integration. Thus, we have the following notation

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \Big[\underbrace{\int_a^b f(x,y)\,dx}_{\text{fun. of } y}\Big]\,dy$$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \Big[\underbrace{\int_c^d f(x,y)\,dy}_{\text{fun. of } x}\Big]\,dx.$$

These integrals are called iterated integrals.

Ex. Evaluate

$$\int_1^3 \int_2^4 (40-2xy)\,dy\,dx.$$

Sol. $\int_1^3 \int_2^4 (40-2xy)\,dy\,dx = \int_1^3 \Big[\int_2^4 (40-2xy)\,dy\Big]\,dx$

$$= \int_1^3 (40y - xy^2)\Big]_2^4 dx$$

$$= \int_1^3 (\underbrace{80 - 12x}_{\text{fun. of } x})\, dx = 80x - 6x^2\Big]_1^3 = 112.$$

<u>Theorem</u>.. Let R be the rectangle defined by $R:\ a \le x \le b,\ c \le y \le d$.

If $f(x,y)$ is continuous on this rectangle, then

$$\iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

$$= \int_a^b \int_c^d f(x,y)\, dy\, dx.$$

<u>Ex</u>. Use a double integral to find the volume of the solid that is bounded <u>above</u> by the plane $z = 4 - x - y$ and <u>below</u> by the rectangle

$R:\ 0 \le x \le 1,\ 0 \le y \le 2.$

<u>Sol</u>. $V = \int_0^2 \int_0^1 (4 - x - y)\, dx\, dy$ or $V = \int_0^1 \int_0^2 (4-x-y)\, dy\, dx$

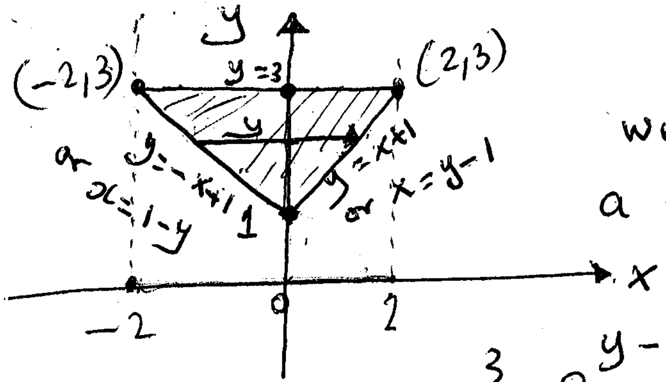
using the first of these, we obtain

Example Evaluate

$$\iint_R (2x - y^2)\, dA$$

over the triangular region between the lines $y = -x + 1$, $y = x + 1$, and $y = 3$.

Sol.



we consider R as a type II.

$$\iint_R (2x - y^2)\, dA = \int_1^3 \int_{1-y}^{y-1} (2x - y^2)\, dx\, dy$$

$$= \int_1^3 \left[x^2 - y^2 x \right]_{1-y}^{y-1} dy$$

$$= \int_1^3 \left[(y-1)^2 - y^2(y-1) - \left((1-y)^2 - y^2(1-y) \right) \right] dy$$

$$= \int_1^3 \left[y^2 - \cancel{2y} + \cancel{1} - y^3 + y^2 - \left(\cancel{1} - \cancel{2y} + \cancel{y^2} - \cancel{y^2} + y^3 \right) \right] dy$$

$$= \int_1^3 \left[2y^2 - 2y^3 \right] dy = \left. \frac{2}{3} y^3 - \frac{y^4}{2} \right]_1^3$$

$$= \frac{-68}{3}.$$

$$V = \int_0^2 \left[\int_0^1 (4-x-y)\,dx \right] dy = \int_0^2 \left[\left(4x - \tfrac{1}{2}x^2 - yx\right) \right]_0^1 dy$$

$$= \int_0^2 \left(4 - \tfrac{1}{2} - y\right) dy = \int_0^2 \left(\tfrac{7}{2} - y\right) dy$$

$$= \tfrac{7}{2}y - \tfrac{1}{2}y^2 \Big]_0^2 = 5 .$$

$$\tfrac{7}{\cancel{2}}(\cancel{2}) - \tfrac{1}{2}(2)^2 \cancel{-0}$$

$$7 - \tfrac{4}{2} = 5$$

<u>Ex.</u> Evaluate

$$\iint_R y^2 x \, dA$$

over the rectangle $R = \{ (x,y) : -3 \le x \le 2, 0 \le y \le$

<u>Sol.</u> $\displaystyle \iint_R y^2 x \, dA = \int_0^1 \int_{-3}^2 y^2 x \, dx \, dy$

$$= \int_0^1 \tfrac{1}{2} y^2 x^2 \Big]_{-3}^2 dy$$

$$= \int_0^1 \left(2y^2 - \tfrac{9}{2}y^2\right) dy = \int_0^1 \tfrac{-5}{2} y^2 \, dy$$

$$= \tfrac{-5}{6} y^3 \Big]_0^1 = \tfrac{-5}{6}$$ (net signed volume).

Ex. Evaluate the integral by choosing a convenient order of integration.

$$\iint_R x\cos(xy)\cos^2\pi x \, dA \;, \quad R = [0,\tfrac{1}{2}] \times [0,\pi]$$

Sol. $\displaystyle\int_0^{1/2}\int_0^{\pi} x\cos(xy)\cos^2\pi x \, dy\, dx$

$$= \int_0^{1/2} \cos^2\pi x \left[\int_0^{\pi} x\cos(xy)\, dy\right] dx$$

$$= \int_0^{1/2} \cos^2\pi x \; \Big[\sin(xy)\Big]_0^{\pi} dx \qquad \underline{\underline{\sin 0 = 0}}$$

$$= \int_0^{1/2} \cos^2\pi x \cdot \sin\pi x \; dx$$

$$= \frac{-1}{\pi}\int_0^{1/2} \cos^2\pi x \cdot (-\pi\sin\pi x)\, dx$$

$$= \frac{-1}{3\pi}\cos^3\pi x\Big]_0^{1/2} = \underbrace{\cancel{\frac{-1}{3\pi}\cos^3\frac{\pi}{2}}}_{\text{Zero}} + \frac{1}{3\pi}\cos^3 0$$

$\cos\frac{\pi}{2} = 0$

$\cos 0 = 1$

$$= \underline{\frac{1}{3\pi}}$$

Ex. $\displaystyle\iint_R x\cos(xy)\cos^2 x \, dA$, $R = [0, \pi/2] \times [0,1]$.

H.W

5/ $\int_0^{\ln 3}\int_0^{\ln 2} e^{x+y}\,dy\,dx = \int_0^{\ln 3} e^{x}\left[\int_0^{\ln 2} e^{y}\,dy\right]dx$

$= \int_0^{\ln 3} e^{x}\left(e^{y}\right]_0^{\ln 2} dx = \int_0^{\ln 3} e^{x}\left(\underline{e^{\ln 2}-e^{0}}\right)dx$

$= \int_0^{\ln 3} e^{x}\,dx = e^{x}\Big]_0^{\ln 3} = 3-1 = 2.$

$2xy$

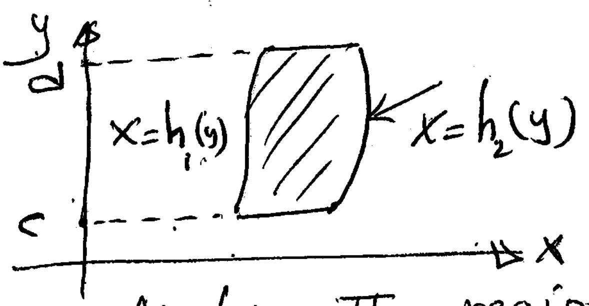
11/ $\int_0^{\ln 2}\int_0^{1} xy\,e^{y^2x}\,dy\,dx = \int_0^{\ln 2}\left[\int_0^{1} xy\,e^{y^2x}\,dy\right]dx$

$= \cancel{\int_0^{\ln 2} \frac{1}{2} e^{y^2x}\,dx} = \int_0^{\ln 2} \frac{1}{2} e^{y^2x}\Big]_0^{1} dx$

$= \int_0^{\ln 2}\left(\frac{1}{2}e^{x}-\frac{1}{2}\right)dx = \frac{1}{2}e^{x}-\frac{1}{2}x\Big]_0^{\ln 2}$

$= \frac{1}{2}e^{\ln 2} - \frac{1}{2}\ln 2 - \left(\frac{1}{2}\right)$

$= 1-\frac{1}{2}\ln 2-\frac{1}{2} = \frac{1}{2}-\frac{1}{2}\ln 2 = \frac{1-\ln 2}{2}.$



A type II region

The following theorem will enable us to evaluate double integrals over type I and type II regions using iterated integrals

Theorem

(a) If R is a type I region on which $f(x,y)$ is continuous, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

(b) If R is a type II region on which $f(x,y)$ is continuous, then

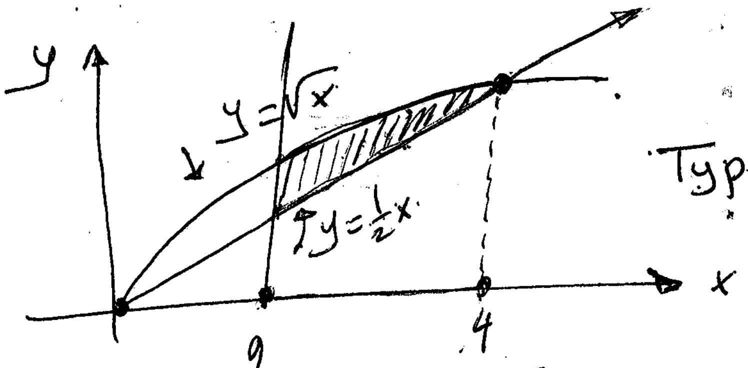
$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

Ex. Evaluate

$$\iint_R xy \, dA$$

over the region R enclosed between $y = \frac{1}{2}x$, $y = \sqrt{x}$, $x = 2$, and $x = 4$.

Sol.



Type I region.

$$\iint_R xy \, dA = \int_2^4 \int_{\frac{1}{2}x}^{\sqrt{x}} xy \, dy \, dx$$

$$= \int_2^4 \frac{xy^2}{2} \Big]_{\frac{1}{2}x}^{\sqrt{x}} dx = \int_2^4 \frac{\cancel{x}}{2}\left(x - \frac{1}{4}x^2\right) dx$$

$$= \int_2^4 \left(\frac{1}{2}x^2 - \frac{1}{8}x^3\right) dx = \frac{1}{6}x^3 - \frac{1}{32}x^4 \Big]_2^4$$

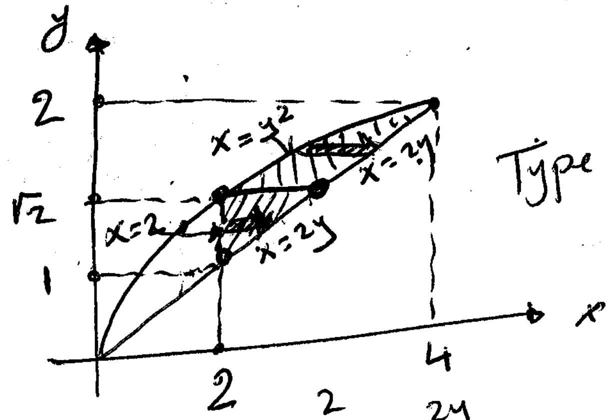
$$= \left(\frac{1}{6}(4)^3 - \frac{1}{32}(4)^4\right) - \left(\frac{1}{6}(2)^3 - \frac{1}{32}(2)^4\right)$$

$$= \frac{64}{6} - \frac{256}{32} - \frac{8}{6} + \frac{16}{32} = \frac{56}{6} - \frac{240}{32}$$

$$= \frac{56}{6} - \frac{15}{2} = \frac{56-45}{6} = \frac{11}{6}.$$

(16')

$$\iint_R xy \, dA ,$$



Type II

$$A = \int_1^{\sqrt{2}} \int_2^{2y} xy \, dx \, dy + \int_{\sqrt{2}}^{2} \int_{y^2}^{2y} xy \, dx \, dy$$

$$= \int_1^{\sqrt{2}} y \, \frac{1}{2} x^2 \Big]_2^{2y} dy + \int_{\sqrt{2}}^{2} \cancel{y} \, \frac{1}{2} x^2 \Big]_{y^2}^{2y} dy$$

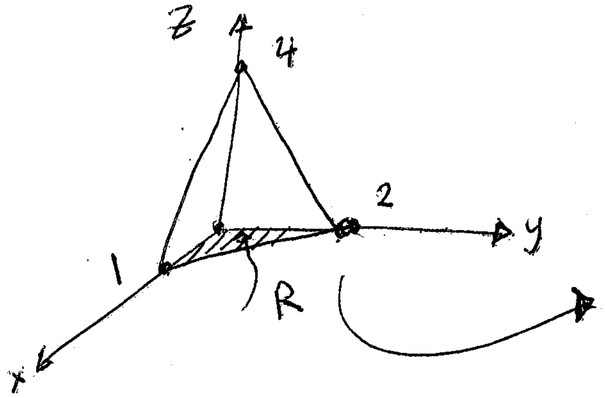
$$= \int_1^{\sqrt{2}} (2y^3 - 2y) \, dy + \int_{\sqrt{2}}^{2} \left(\frac{1}{2} y^5 - 2y^3\right) dy$$

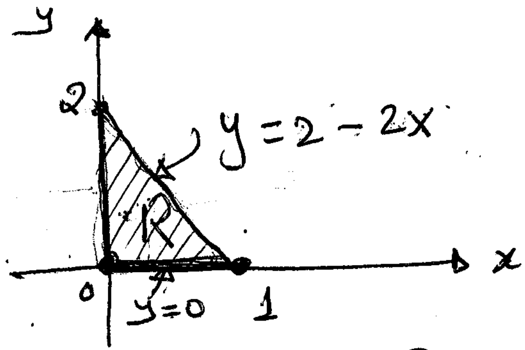
$$= \frac{1}{2} y^4 - y^2 \Big]_1^{\sqrt{2}} + \frac{1}{12} y^6 - \frac{1}{2} y^4 \Big]_{\sqrt{2}}^{2}$$

$$= \frac{11}{6} .$$

Ex. Use a double integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane $z = 4 - 4x - 2y$.

Sol. To find the region R, set $z = 0$, we get line $4 - 4x - 2y = 0$, or $y = 2 - 2x$, so the region R is bounded by the x-axis, the y-axis, and the line $y = 2 - 2x$,





we consider R as a type I.

$$\Rightarrow V = \iint_R (4 - 4x - 2y)\, dA = \int_0^1 \int_0^{2-2x} (4 - 4x - 2y)\, dy\, dx$$

$$= \int_0^1 \left(4y - 4xy - y^2\right)\Big]_0^{2-2x} dx$$

$$= \int_0^1 \left[4(2-2x) - 4x(2-2x) - (2-2x)^2\right] dx$$

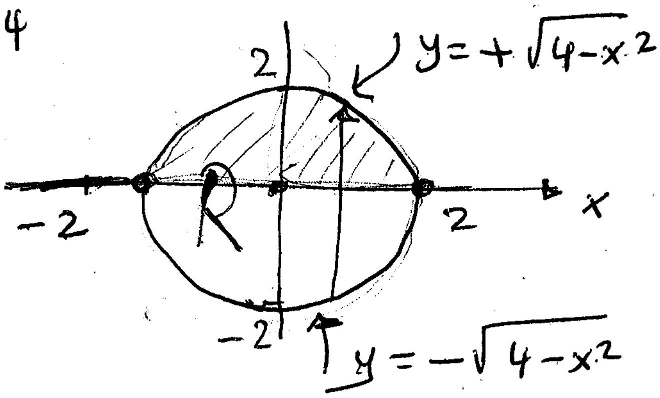
$$= \int_0^1 (4 - 8x + 4x^2)\, dx = 4x - 4x^2 + \frac{4}{3}x^3\Big]_0^1$$

$$= \frac{4}{3}.$$

Ex. Find the volume of the solid bounded by the cylinder $x^2+y^2=4$ and the planes $z+y=4$ and $\underbrace{z=0}_{xy\text{-plane}}$.

Sol. The region R lies within the circle

$$x^2+y^2=4$$



Consider the region R as a type I.

$$\Rightarrow \quad V=\iint_R (4-y)\,dA=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(4-y)\,dy\,dx$$

$$=\int_{-2}^{2}\left[4y-\frac{1}{2}y^2\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}dx$$

$$=\int_{-2}^{2}\left[\left\{4\sqrt{4-x^2}-\cancel{\tfrac{1}{2}(4-x^2)}\right\}-\left\{-4\sqrt{4-x^2}-\cancel{\tfrac{1}{2}(4-x^2)}\right\}\right]dx$$

$$=\int_{-2}^{2}8\sqrt{4-x^2}\,dx=8\,[\text{area of the upper semicircle of radius 2 centered at the origin}]$$

$$=8\left[\frac{4\pi}{2}\right]=16\pi.$$

Reversing The Order of Integration

Sometimes the evaluation of an iterated integral can be simplified by reversing the order of integration as illustrated by the following example.

Ex. Evaluate

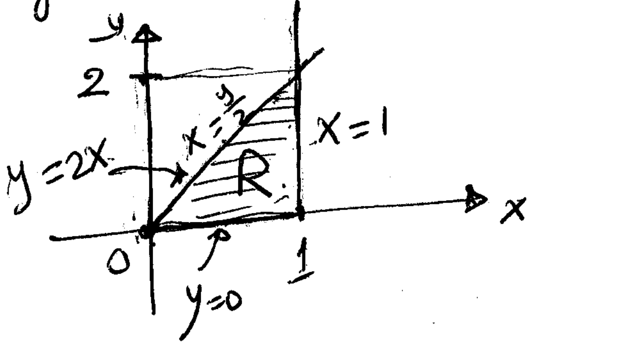
$$\int_0^2 \int_{y/2}^1 e^{x^2}\,dx\,dy$$

Sol. We cannot integrate the inside integral, since there is no elementary antiderivative of e^{x^2}. Thus we must reverse the order of integration. To do so, we should determin the region R as follows,

$$\frac{y}{2} \le x \le 1 \ ; \ i.e.,$$

the values of x varies from the line $x = \frac{y}{2}$ to the line $x = 1$. $\boxed{y = 2x}$

and the values of y varies from 0 to 2.



$$\Rightarrow \int_0^2 \int_{y/2}^1 e^{x^2}\,dx\,dy = \iint_R e^{x^2}\,dA$$

$$= \int_0^1 \int_0^{2x} e^{x^2}\, dy\, dx = \int_0^1 e^{x^2} \int_0^{2x} dy\, dx$$

$$= \int_0^1 e^{x^2} [y]_0^{2x}\, dx = \int_0^1 2x\, e^{x^2}\, dx$$

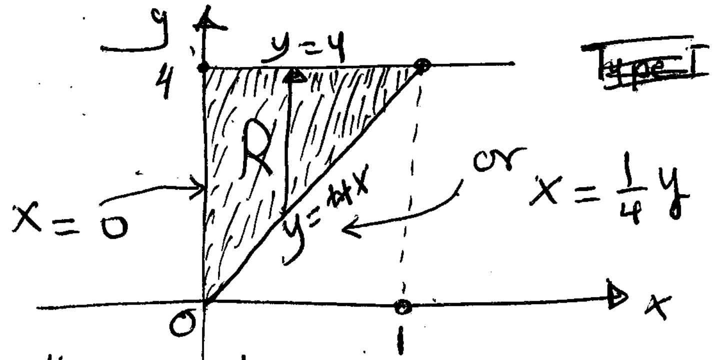
$$= e^{x^2}\Big]_0^1 = e - 1. \qquad , \quad \int e^u du = e^u + C$$

Ex. $\displaystyle\int_0^1 \int_{4x}^4 e^{-y^2}\, dy\, dx$

Sol. Reversing the order of integration (why?)

$$4x \le y \le 4 \quad \text{and} \quad 0 \le x \le 1$$

the values of y varies from the line $y = 4x$ to the line $y = 4$, and the values of x varies from 0 to 1.



$$\Rightarrow \int_0^1 \int_{4x}^4 e^{-y^2}\, dy\, dx$$

$$= \iint_R e^{-y^2}\, dA = \int_0^4 \int_0^{\frac{1}{4}y} e^{-y^2}\, dx\, dy$$

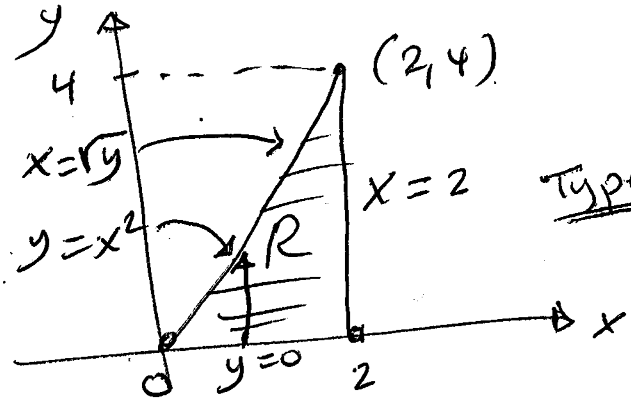
$$= \int_0^4 e^{-y^2}\, x\Big]_0^{y/4} dy = \int_0^4 \frac{1}{4}\, y\, e^{-y^2}\, dy$$

$$= \frac{-1}{8} e^{-y^2} \Big]_0^4 = \frac{-1}{8}\left[e^{-16} - 1\right] = \frac{1-e^{-16}}{8}.$$

<u>Example</u> Evaluate $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\,dx\,dy$.

<u>Sol.</u> Reversing the order of integration (Why?)

Since, the values of x varies from $x=\sqrt{y}$ $(y=x^2)$ to $x=2$ and the values of y varies from 0 to 4, the region R of integration as shown below



Type II

$$\Rightarrow \int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\,dx\,dy$$

$$= \iint_R e^{x^3}\,dA = \int_0^2 \int_0^{x^2} e^{x^3}\,dy\,dx \quad \text{Type I}$$

$$= \int_0^2 e^{x^3} (y]_0^{x^2}\,dx = \int_0^2 x^2 e^{x^3}\,dx$$

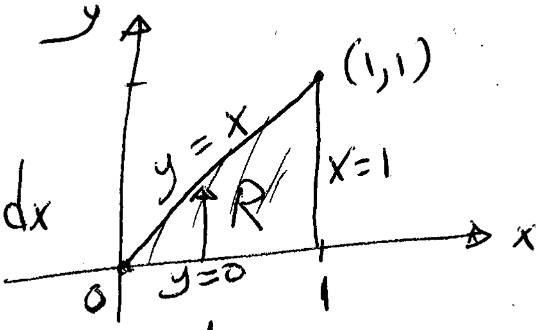
$$= \frac{1}{3} e^{x^3}\Big]_0^2 = \frac{1}{3}\left[e^8 - 1\right]$$

Ex. Evaluate

$$\iint_R \frac{\sin x}{x}\, dA,$$

where R is the triangle in the xy-plane bounded by the x-axis, the line $y=x$, and the line $x=1$.

Sol.



$$\iint_R \frac{\sin x}{x}\, dA = \int_0^1 \int_0^x \frac{\sin x}{x}\, dy\, dx$$

$$= \int_0^1 \frac{\sin x}{x} \cdot \left(y \right]_0^x dx = \int_0^1 \frac{\sin x}{\cancel{x}} \cdot \cancel{x}\, dx$$

$$= -\cos x \Big]_0^1 = -\cos 1 + 1.$$

Area Calculated As A Double Integral

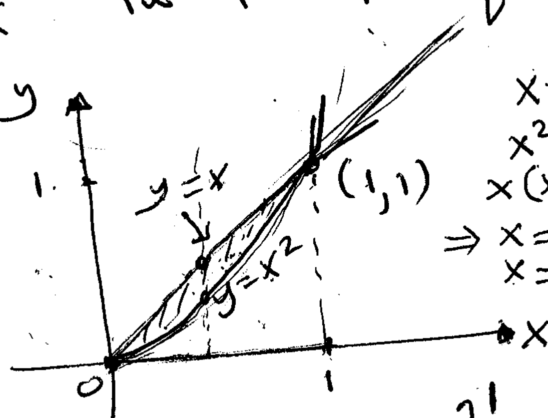
If we take $f(x,y) = 1$ in the definition of the double integral (formula (1)), over a region R, then the total volume of a solid S becomes

$$V \approx \sum_{k=1}^{n} V_k = \sum_{k=1}^{n} \Delta A_k$$

As Δx and Δy approach zero, the converge of R by the ΔA_k's becomes increasingly complete and we define the area of R to be the limit

$$\text{Area}(R) = \lim_{\Delta A \to 0} \sum_{k=1}^{n} \Delta A_k = \iint_R dA .$$

Ex. Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadran



$x = x^2$
$x^2 - x = 0$
$x(x-1) = 0$
$\Rightarrow x = 0,$
$x = 1$

Sol.

$$A = \int_0^1 \int_{x^2}^{x} dy\, dx \qquad \text{Type I}$$

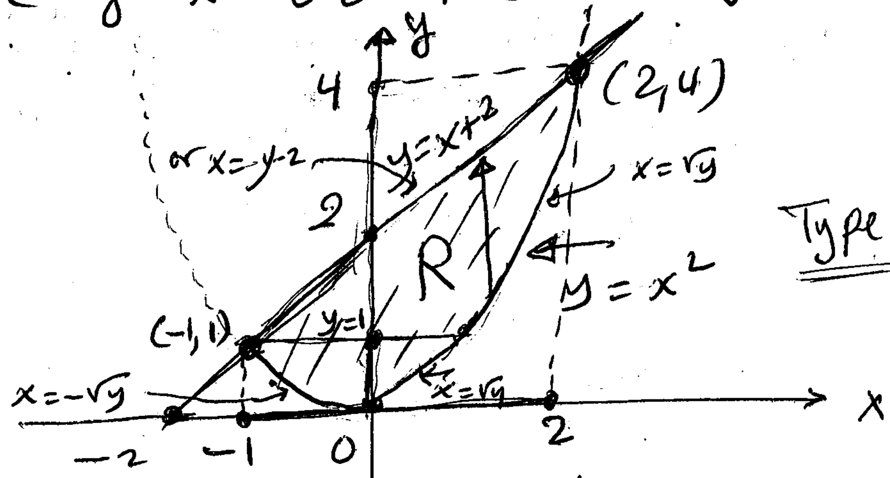
$$= \int_0^1 [y]_{x^2}^{x}\, dx = \int_0^1 (x - x^2)\, dx = \frac{1}{2}x^2 - \frac{1}{3}x^3 \Big]_0^1 = \frac{1}{6}.$$

$$\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

or Type II $\displaystyle \int_0^1 \int_y^{\sqrt{y}} dx\, dy = \frac{1}{6}$

Ex. Find the area of the region R enclosed by the curve $y = x^2$ and the line $y = x+2$.

Sol.



Type I

To find the points of intersection, solve

$$x^2 = x+2 \quad \text{or} \quad x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0 \Rightarrow x = 2,\ x = -1$$

$$\Rightarrow A = \int_{-1}^{2} \int_{x^2}^{x+2} dy\,dx = \int_{-1}^{2} (y]_{x^2}^{x+2}\,dx$$

Type I

$$= \int_{-1}^{2} (x+2-x^2)\,dx = \tfrac{1}{2}x^2 + 2x - \tfrac{1}{3}x^3\Big]_{-1}^{2}$$

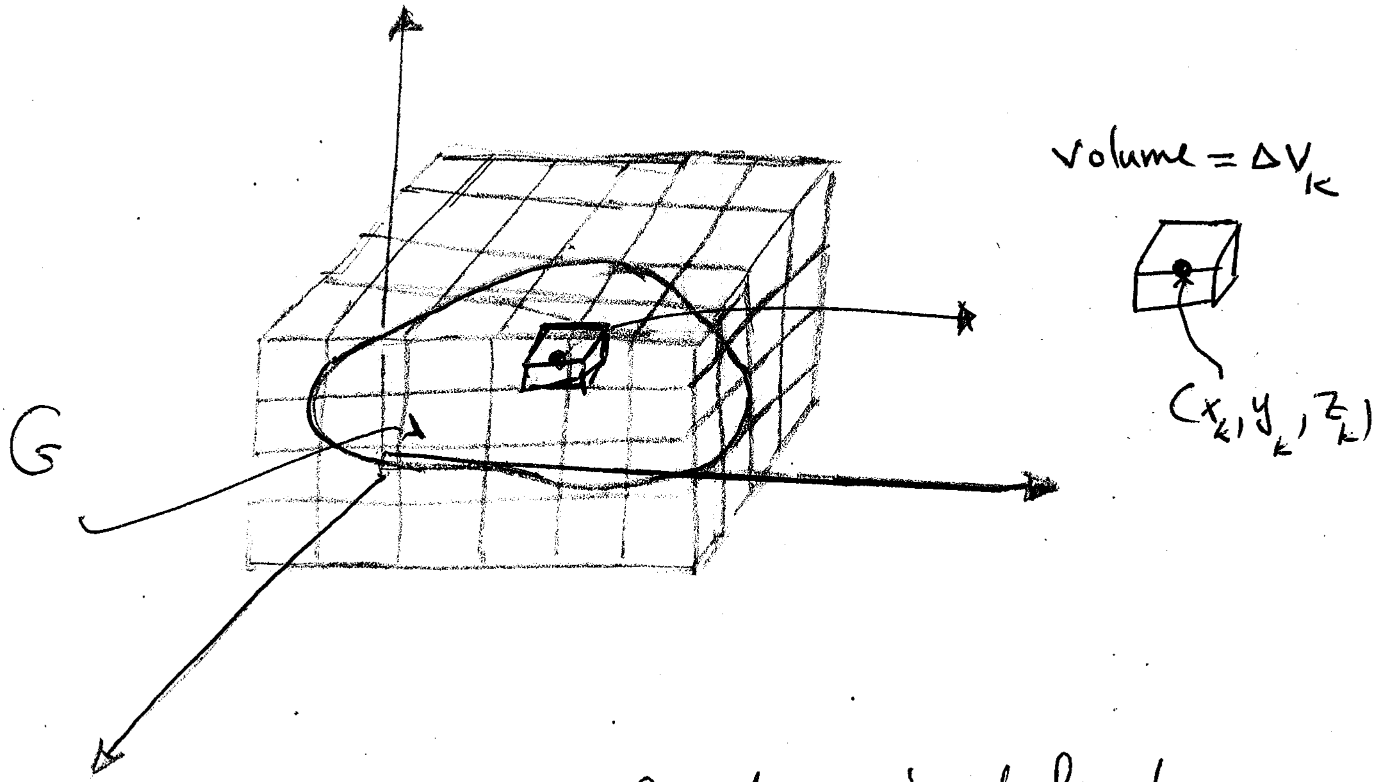
$$= \frac{9}{2}.$$

Type II $$A = \int_{0}^{1} \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_{1}^{4} \int_{y-2}^{\sqrt{y}} dx\,dy$$

$$= \frac{4}{3} + \frac{19}{6} = \frac{9}{2}.$$

Triple Integrals

Let G be a finite solid, enclosed in a suitably large box whose sides are parallel to the coordinate planes.



Let $f(x,y,z)$ be a function is defined over G in an xyz-coordinate system. To define the triple integral of $f(x,y,z)$ over G, we divide the box into n subboxes by planes parallel to the coordinate planes, as shown above, and discard those subboxes that contain any points outside of G and choose an arbitrary point ~~in~~ each of the remaining subboxes. Let the volume of the kth subbox is denoted by ΔV_k and the selected point in the kth subbox by (x_k, y_k, z_k). Next, we form the product

$$f(x_k, y_k, z_k)\,\Delta V_k$$

for each subbox. Adding all the products for all the subboxes to obtain the <u>Riemann sum</u>

$$\sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta V_k \,.$$

If we let $n \to \infty$, then the length, width, and height approach ~~zero~~. The limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta V_k = \iiint_G f(x,y,z)\, dV$$

is called the triple integral of $f(x,y,z)$ over the region G.

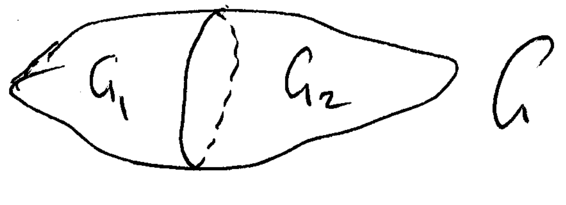
Properties of Triple Integrals

(1) $\iiint_G c\, f(x,y,z)\, dV = c \iiint_G f(x,y,z)\, dV$ (c a con

(2) $\iiint_G \left[f(x,y,z) + g(x,y,z) \right] dV = \iiint_G f(x,y,z)\, dV + \iiint_G g(x,$

(3) $\iiint_G \left[f(x,y,z) - g(x,y,z) \right] dV = \iiint_G f(x,y,z)\, dV - \iiint_G g(x,y,z)\, d$

(4) $\iiint_G f(x,y,z)\, dV = \iiint_{G_1} f(x,y,z)\, dV + \iiint_{G_2} g(x,y,z)\, dV$



(22

Theorem. Let G be the rectangular box defined by the inequalities

$$a \le x \le b, \quad c \le y \le d, \quad k \le z \le l.$$

If f is continuous on the region G, then

$$\iiint_G f(x,y,z)\,dV = \int_a^b \int_c^d \int_k^l f(x,y,z)\,dz\,dy\,dx$$

Example Evaluate

$$\iiint_G 12xy^2z^3\,dV$$

where G is defined by $-1 \le x \le 2$, $0 \le y \le 3$, $0 \le z \le 2$.

Sol.

$$\iiint_G 12xy^2z^3\,dV = \int_{-1}^{2}\int_0^3\int_0^2 12xy^2z^3\,dz\,dy\,dx$$

$$= \int_{-1}^{2}\int_0^3 \cancel{12}^{3} x y^2 \left(\tfrac{1}{\cancel{4}} z^4\right)\Big]_0^2 dy\,dx$$

$$= \int_{-1}^{2}\int_0^3 48\,x\,y^2\,dy\,dx = \int_{-1}^{2} 48x\left(\tfrac{1}{3}y^3\right)\Big]_0^3 dx$$

$$= 432\int_{-1}^{2} x\,dx = 432\left(\tfrac{1}{2}x^2\right]_{-1}^{2} = 648.$$

Example Evaluate

$$\iiint_G xy \sin yz \, dV,$$ where G is the rectangular box defined by:

$$0 \le x \le \pi, \quad 0 \le y \le 1, \quad 0 \le z \le \pi/6$$

Sol.

$$\iiint_G xy \sin yz \, dV = \int_0^{\pi}\int_0^{1}\int_0^{\pi/6} xy \sin yz \, dz \, dy \, dx$$

$$= \int_0^{\pi}\int_0^{1} x \left(-\cos yz\right]_0^{\pi/6} dy \, dx$$

$$= \int_0^{\pi}\int_0^{1} x \left(-\cos \frac{\pi y}{6} + 1\right) dy \, dx$$

$$= \int_0^{\pi}\int_0^{1} \left(x - x \cos \frac{\pi y}{6}\right) dy \, dx$$

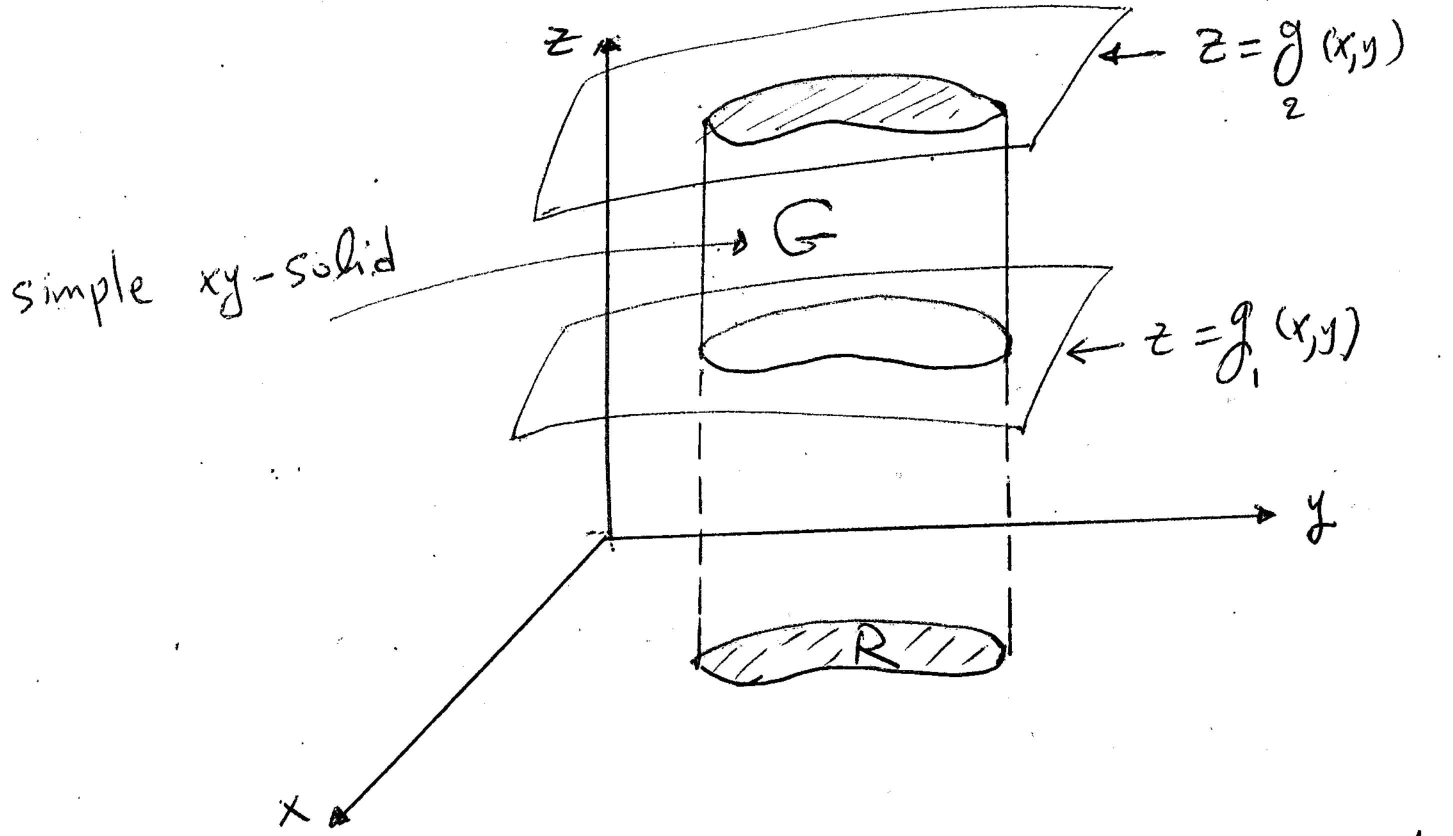
$$= \int_0^{\pi} \left(xy - \frac{6x}{\pi} \sin \frac{\pi y}{6}\right]_0^{1} dx$$

$$= \int_0^{\pi} \left(x - \frac{6x}{\pi} \sin \frac{\pi}{6}\right) dx = \left.\frac{1}{2}x^2 - \frac{3}{2\pi}x^2\right]_0^{\pi}$$

$$= \pi\left(\frac{\pi - 3}{2}\right).$$

Evaluating Triple Integrals over More General Regions

Let G be a solid that is bounded above by a surface $z = g_2(x,y)$ and below by a surface $z = g_1(x,y)$ and that the projection of the solid on the xy-plane is a type I or type II region R (see the figure). In addition, we will assume that $g_1(x,y)$ and $g_2(x,y)$ are continuous on R and that $g_1(x,y) \leq g_2(x,y)$ on R. A solid of this type is called a <u>simple xy-solid</u>.



The following theorem will enable us to evaluate triple integrals over simple xy-solids.

<u>Theorem</u> Let G be a simple xy-solid with upper surface $z = g_2(x,y)$ and lower surface $z = g_1(x,y)$, and let R be the projection of G on the xy-plane. If $f(x,y,z)$ is continuous on G, then.

$$\iiint_G f(x,y,z)\, dV = \iint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\, dz \right] dA$$

<u>Note</u> If R is a type I region, then

$$\iint_R f(x,y)\, dA = \int_{x=a}^{x=b} \int_{y=h_1(x)}^{y=h_2(x)} f(x,y)\, dy\, dx$$

and If R is a type II region, then

$$\iint_R f(x,y)\, dA = \int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\, dx\, dy.$$

i.e.,

$$\iiint_G f(x,y,z)\, dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\, dz \right] dy\, dx$$

(Type I).

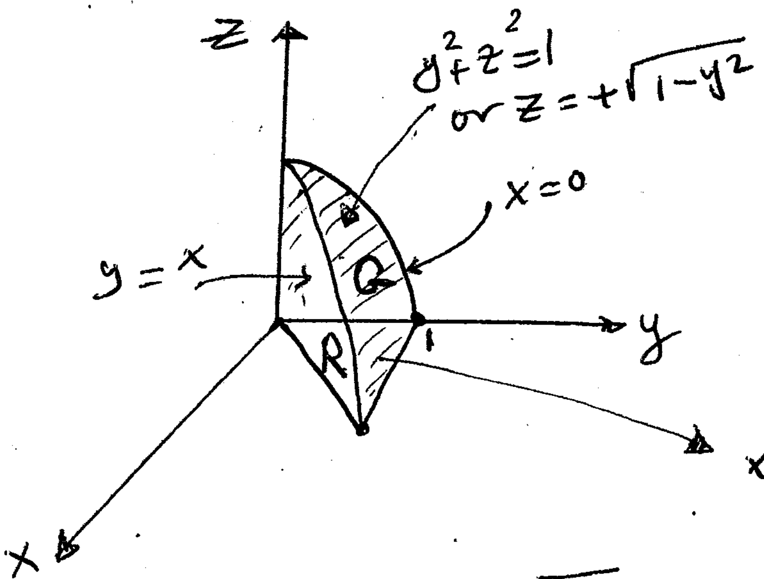
or

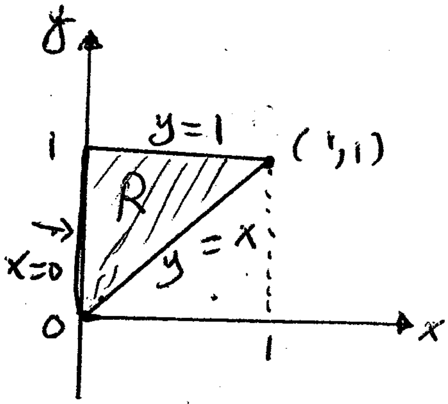
$$\iiint_G f(x,y,z)\, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\, dz \right] dx\, dy$$

Ex Let G be the wedge in the first octant (24) that is cut from the cylindrical solid $y^2+z^2 \le 1$ by the plane $y = x$ and $x = 0$. Evaluate

$$\iiint_G z \, dV$$

Sol.





$$\iiint_G z \, dV = \iint_R \left[\int_0^{\sqrt{1-y^2}} z \, dz \right] dA$$

$$= \int_0^1 \int_x^1 \left[\int_0^{\sqrt{1-y^2}} z \, dz \right] dy \, dx$$

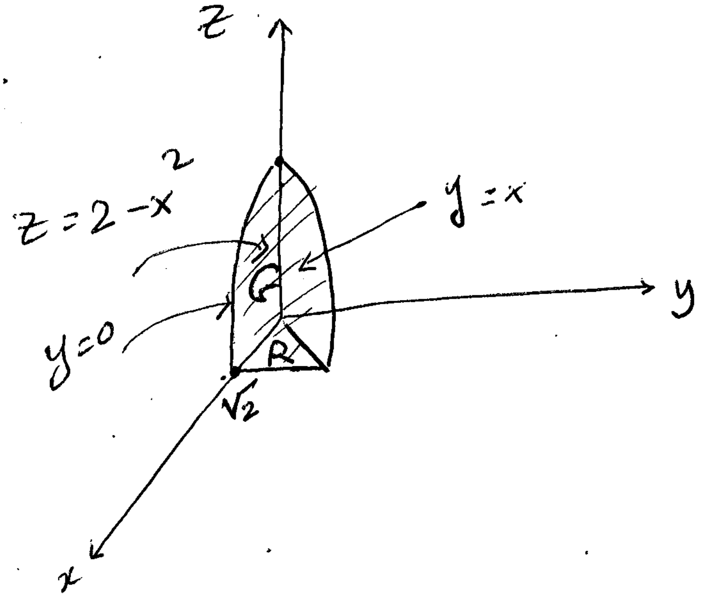
$$= \int_0^1 \int_x^1 \frac{1}{2} z^2 \Big]_0^{\sqrt{1-y^2}} dy \, dx = \frac{1}{2} \int_0^1 \int_x^1 (1-y^2) \, dy \, dx$$

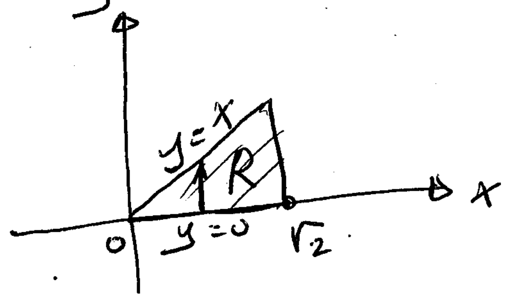
$$= \frac{1}{2} \int_0^1 \left[y - \frac{1}{3} y^3 \right]_x^1 dx = \frac{1}{2} \int_0^1 \left[\frac{2}{3} - x + \frac{1}{3} x^3 \right] dx$$

$$= \frac{1}{2} \left[\frac{2}{3} x - \frac{1}{2} x^2 + \frac{1}{12} x^4 \right]_0^1 = \frac{1}{8}$$

Ex. Evaluate $\iiint_G xyz \, dV$, where G is the solid in the first octant that is bounded by the parabolic cylinder $z = 2 - x^2$ and the planes $z = 0$, $y = x$, and $y = 0$.

sol.





$$\iiint_G xyz \, dV$$

$$= \iint_R \left[\int_{z=0}^{z=2-x^2} xyz \, dz \right] dA$$

$$= \int_0^{\sqrt{2}} \int_0^{x} \left[\int_{z=0}^{z=2-x^2} xyz \, dz \right] dy \, dx$$

$$= \int_0^{\sqrt{2}} \int_0^{x} \frac{1}{2} xy \, z^2 \Big]_0^{2-x^2} dy\,dx = \int_0^{\sqrt{2}} \int_0^{x} \frac{1}{2} x (2-x^2)^2 y \, dy \, dx$$

$$= \int_0^{\sqrt{2}} \frac{1}{4} x (4 - 4x^2 + x^4) \left(y^2 \Big|_0^x \right) dx = \frac{1}{4} \int_0^{\sqrt{2}} [4x^3 - 4x^5 + x^7] \, dx$$

$$= \frac{1}{6}.$$

Note

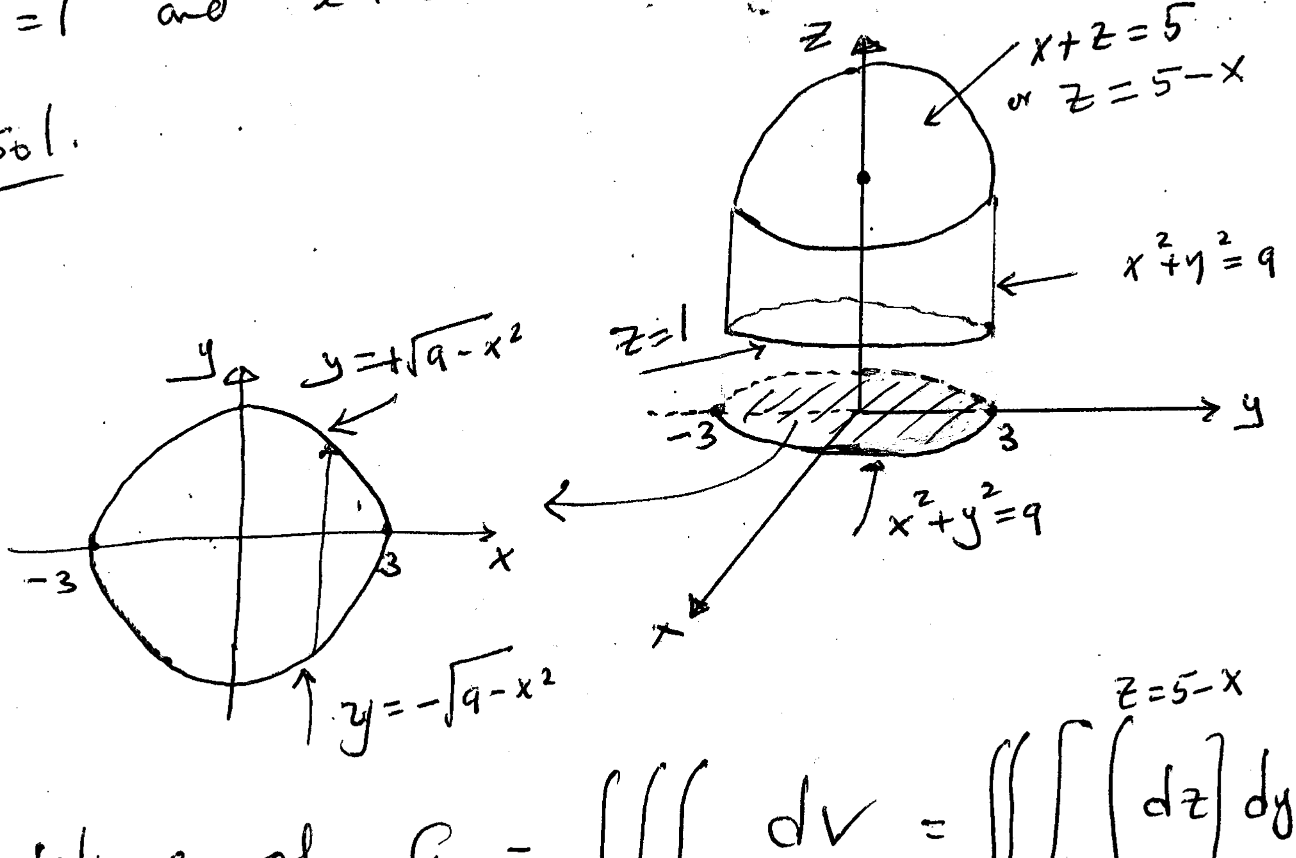
When $f(x,y,z)=1$, formula (1) becomes

$$\lim_{n\to\infty} \sum_{k=1}^{n} \Delta V_k = \iiint_G dV$$

$$\Rightarrow \text{Volume of } G = \iiint_G dV$$

Ex. Use triple integral to find the volume of the solid within the cylinder $x^2+y^2=9$ and between $z=1$ and $x+z=5$.

Sol.



$$\text{Volume of } G = \iiint_G dV = \iint_R \left[\int_{z=1}^{z=5-x} dz\right] dy\,dx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\left[\int_{z=1}^{z=5-x} dz\right] dy\,dx = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4-x)\,dy\,dx$$

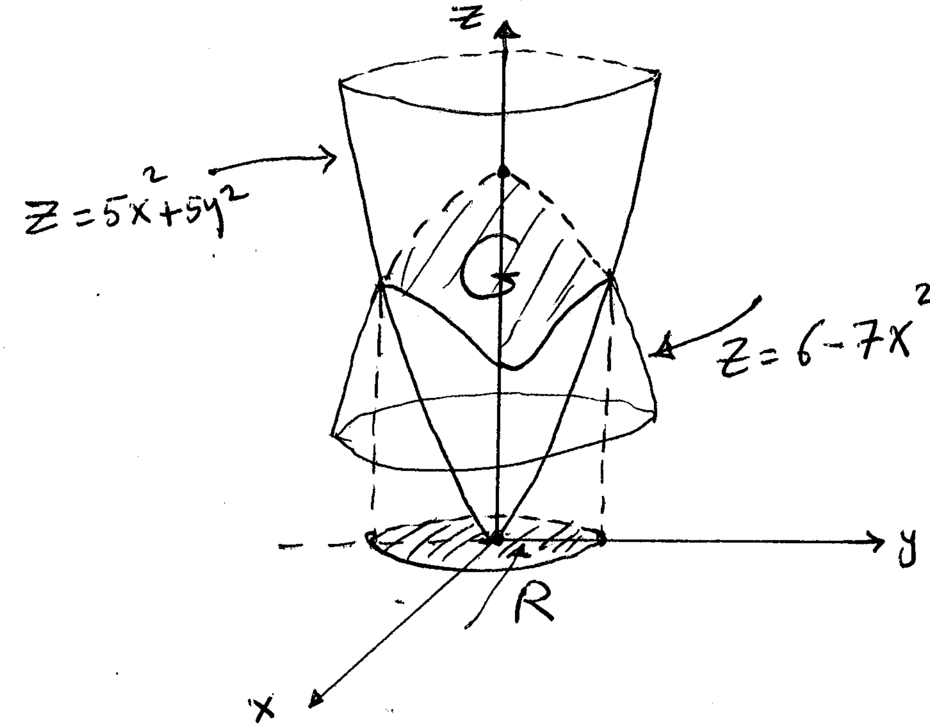
$$= \int_{-3}^{3}(4-x)\left[2\sqrt{9-x^2}\right]dx = \int_{-3}^{3} 8\sqrt{9-x^2}\,dx - \int_{-3}^{3} \underbrace{2x\sqrt{9-x^2}}_{\text{odd fun.}}\,dx$$

$$= 8\left[\tfrac{9\pi}{2}\right] - \text{Zero} = 36\pi.$$

Ex. Find the volume of the solid enclosed between the paraboloids

$$Z = 5x^2 + 5y^2 \quad \text{and} \quad Z = 6 - 7x^2 - y^2$$

sol.

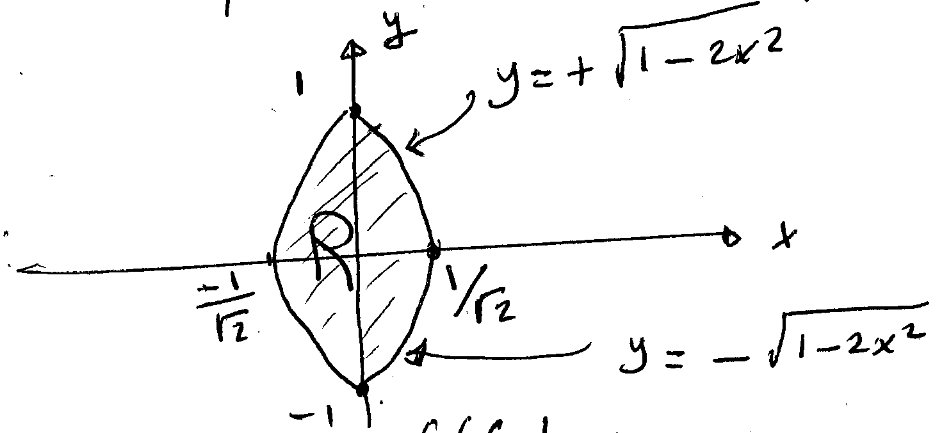


To find the projection R, solving the equation

$$5x^2 + 5y^2 = 6 - 7x^2 - y^2$$

$$12x^2 + 6y^2 = 6 \Rightarrow \boxed{2x^2 + y^2 = 1}$$

which is an ellipse.



$\Rightarrow$ Volume of $G = \iiint_G dV$

$$= \iint_R \left[\int_{5x^2+5y^2}^{6-7x^2-y^2} dz \right] dA$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} \int_{5x^2+5y^2}^{6-7x^2-y^2} dz \; dy \; dx$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} \left[6-7x^2-y^2-5x^2-5y^2 \right] dy \, dx$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} \left[6-12x^2-6y^2 \right] dy \, dx$$

$$= 6 \int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1-2x^2)y - \frac{1}{3}y^3 \Big]_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} dx$$

$$= 6 \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \Big[(1-2x^2)\sqrt{1-2x^2} - \frac{1}{3}\left(\sqrt{1-2x^2}\right)^3 - \Big((1-2x^2)(-\sqrt{1-2x^2}) - \frac{1}{3}\left(-\sqrt{1-2x^2}\right)^3 \Big] dx$$

$$= 6 \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{4}{3}(1-2x^2)^{3/2} dx = 8 \int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1-2x^2)^{3/2} dx$$

Let $x = \frac{1}{\sqrt{2}} \sin\theta \Rightarrow 1-2x^2 = 1-\sin^2\theta = \cos^2\theta$

and $dx = \frac{1}{\sqrt{2}} \cos\theta \, d\theta$.

$$\Rightarrow 8\int(1-2x^2)^{1/2}dx = \frac{8}{\sqrt{2}}\int \cos^4\theta\, d\theta$$

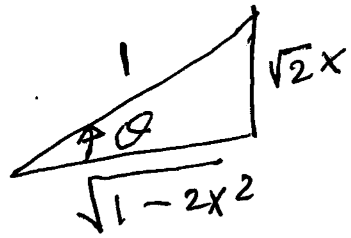
$$= \frac{8}{\sqrt{2}}\int(\cos^2\theta)^2 d\theta = \frac{8}{\sqrt{2}}\int\left(\frac{1+\cos2\theta}{2}\right)^2 d\theta$$

$$= \frac{2}{\sqrt{2}}\int\left(1+2\cos2\theta+\cos^2 2\theta\right)d\theta$$

$$= \sqrt{2}\int\left(1+2\cos2\theta+\frac{1+\cos4\theta}{2}\right)d\theta$$

$$= \sqrt{2}\int\left(\frac{3}{2}+2\cos2\theta+\frac{1}{2}\cos4\theta\right)d\theta$$

$$= \sqrt{2}\left[\left(\frac{3}{2}\theta+\sin2\theta+\frac{1}{8}\sin4\theta\right]+c.$$



$$\because \sin\theta = \sqrt{2}x \Longrightarrow$$

$$\cos\theta = \sqrt{1-2x^2}$$

$$\sin2\theta = 2\sin\theta\cos\theta = 2\sqrt{2}x\sqrt{1-2x^2}$$

$$\frac{1}{8}\sin4\theta = \frac{1}{8}\left[2\sin2\theta\cos2\theta\right]$$

$$= \frac{1}{2\cancel{4}}\left[\cancel{2}\sqrt{2}x\sqrt{1-2x^2}\left((1-2x^2)-2x^2\right)\right]$$

$$\Rightarrow 8\int_{-1/\sqrt{2}}^{1/\sqrt{2}}(1-2x^2)^{3/2}dx = \sqrt{2}\left[\frac{3}{2}\sin^{-1}\sqrt{2}x+2\sqrt{2}x\sqrt{1-2x^2}+\frac{1}{2}\left(\sqrt{2}x\sqrt{1-2x^2}\{(1-4x^2)\}\right]_{-1/\sqrt{2}}^{1/\sqrt{2}}$$

$$= \frac{\sqrt{2}}{2}(3)\left[\sin^{-1}(1)-\sin^{-1}(-1)\right] = \frac{3}{\sqrt{2}}(2\sin^{-1}1)$$

$$= \frac{3}{\sqrt{2}}\left(\cancel{2}\cdot\frac{\pi}{\cancel{2}}\right) = \boxed{\frac{3\pi}{\sqrt{2}}}$$

Exercises,

Evaluate

(1) $$\int_1^2 \int_z^2 \int_0^{\sqrt{3}\,y} \frac{y}{x^2+y^2}\, dx\, dy\, dz$$

$$= \int_1^2 \int_z^2 \cancel{y}\, \frac{1}{\cancel{y}} \tan^{-1}\frac{x}{y} \Big]_0^{\sqrt{3}\,y} dy\, dz$$

$$= \int_1^2 \int_z^2 \tan^{-1}\sqrt{3}\; dy\, dz$$

$$= \int_1^2 \int_z^2 \frac{\pi}{3}\, dy\, dz = \frac{\pi}{3}\int_1^2 (2-z)\, dz$$

$$= \frac{\pi}{3}\left[2z - \frac{1}{2}z^2\right]_1^2 = \frac{\pi}{3}\left(2-\frac{3}{2}\right) = \frac{\pi}{6}.$$

(2) $\iiint_G \cos\left(\frac{z}{y}\right) dV$, where G is the solid defined by the inequalities $\frac{\pi}{6} \le y \le \frac{\pi}{2}$, $y \le x \le \frac{\pi}{2}$, $0 \le z \le xy$

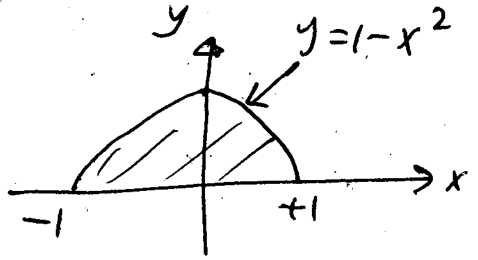
Sol. $$\iiint_G \cos\left(\frac{z}{y}\right) dV = \int_{\pi/6}^{\pi/2} \int_y^{\pi/2} \int_0^{xy} \cos\left(\frac{z}{y}\right) dz\, dx\, dy$$

$$= \int_{\pi/6}^{\pi/2} \int_y^{\pi/2} y \sin\left(\frac{z}{y}\right) \Big]_0^{xy} dx\, dy = \int_{\pi/6}^{\pi/2} \int_y^{\pi/2} y \underbrace{\sin x\, dx}\, dy$$

$$= \int_{\pi/6}^{\pi/2} -y \cos x \Big]_y^{\pi/2} dy = \int_{\pi/6}^{\pi/2} \underbrace{y \cos y\, dy}_{u=y} = \frac{5\pi - 6\sqrt{3}}{12}$$

$u = y$, $dv = \cos y\, dy$

Ex. Evaluate $\iiint_G y\,dV$, where G is the sol enclosed by the plane $z=y$, the $\underbrace{xy\text{-plane}}_{z=0}$, and the parabolic cylinder $y=1-x^2$



Sol.

$$\iiint_G y\,dV = \iint_R \left[\int_0^y y\,dz\right] dA$$

$$= \int_{-1}^{1}\int_0^{1-x^2}\int_0^y y\,dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_0^{1-x^2} y^2\,dy\,dx = \int_{-1}^{1} \left.\frac{1}{3}y^3\right]_0^{(1-x^2)} dx$$

$$= \frac{1}{3}\int_{-1}^{1}(1-x^2)^3\,dx$$

Note $(x+y)^3 = x^3+3x^2y+\frac{6xy^2}{2}+\dots$
$= x^3+3x^2y+3xy^2+y^3$

$(x-y)^3 = x^3-3x^2y+3xy^2-y^3$

$$= \frac{2}{3}\int_0^1 (1-x^2)^3\,dx$$

$$= \frac{2}{3}\int_0^1 (1-3x^2+3x^4-x^6)\,dx$$

$$= \frac{2}{3}\left[x-x^3+\frac{3}{5}x^5-\frac{1}{7}x^7\right]_0^1 = \frac{2}{3}\left[\frac{3}{5}-\frac{1}{7}\right]$$

$$= \frac{32}{105}.$$

Answers

1. (a) $3x^2$, $3x^2$, $3x^2$ (b) Whatever the constant, it is zero when differentiated.

2. Any constant will disappear (i.e. become zero) when differentiated so one must be reintroduced to reverse the
process.

2. A table of integrals

We could use a table of derivatives to find integrals, but the more common ones are usually found in a 'Table of Integrals' such as that shown below. You could check the entries in this table using your knowledge of differentiation. Try this for yourself.

Table 1: Integrals of Common Functions

function $f(x)$	indefinite integral $\int f(x)\,dx$
constant, k	$kx + c$
x	$\frac{1}{2}x^2 + c$
x^2	$\frac{1}{3}x^3 + c$
x^n	$\dfrac{x^{n+1}}{n+1} + c, \quad n \neq -1$
x^{-1} (or $\dfrac{1}{x}$)	$\ln\|x\| + c$
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$
$\cos kx$	$\dfrac{1}{k}\sin kx + c$
$\sin kx$	$-\dfrac{1}{k}\cos kx + c$
$\tan kx$	$\dfrac{1}{k}\ln\|\sec kx\| + c$
e^x	$e^x + c$
e^{-x}	$-e^{-x} + c$
e^{kx}	$\dfrac{1}{k}e^{kx} + c$

When dealing with the trigonometric functions the variable x must always be measured in radians and not degrees. Note that the fourth entry in the Table, for x^n, is valid for any value of n, positive or negative, whole number or fractional, *except* $n = -1$. When $n = -1$ use the fifth entry in the Table.