Al-Mansour University College

قسم الهندسة المدنية
المرحلة الثانية

Civil Eng. Dept.
2nd. Stage

Engineering Mathematics

2022 - 2023

Lec.2

الرياضيات

Matrices And Determinants ①

المصفوفات والمحددات

Defn. A rectangular array of real (or complex) numbers of the form :

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \quad \cdots (1)$$

is called a matrix. (المصفوفة)

- The numbers $a_{11}, a_{12}, \cdots, a_{mn}$ are called the elements of the matrix.
- The horizontal Lines are called rows (صفوف)
- The vertical Lines are called Columns (أعمدة)
- A matrix with m rows and n column is called $m \times n$ matrix (read m by n matrix).
- Matrices will be denoted by capital bold-faced Letters $A, B, \ldots$ or by $[a_{jk}], [b_{jk}], \cdots$
- A matrix $[a_1 \ a_2 \ \cdots \ a_n]$ having only one row is called a Row Matrix or row vector.
- A vector having only one column is called a column matrix or column vector. $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

- The elements of a matrix may be real or complex numbers. If all the elements of a matrix are real the matrix is called a Real Matrix (مصفوفة حقيقية).

- A matrix having the same number of rows and columns is called a square matrix (مصفوفة مربعة), and the number of rows is called its order. A square matrix of order n is called an $n \times n$ matrix (read n by n matrix).

The Types of Matrices انواع المصفوفات

① Square Matrix: المصفوفة المربعة

number of rows = number of columns

EX.

$$\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} , \quad \begin{bmatrix} -1 & 3 & 2 \\ 0 & 7 & 1 \\ 2 & 4 & 5 \end{bmatrix}$$

عدد الصفوف = عدد الاعمدة = 2

عدد الصفوف = عدد الاعمدة = 3

② Diagonal Matrix المصفوفة القطرية

It is square matrix, all the elements are zero except the elements of the main diagonal are not zero.

EX.

$$\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} , \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

③ Triangular Matrix المصفوفة المثلثية

It is a square matrix whose elements upper or lower

EX.

$$\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} , \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} , \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 5 & 0 \\ 4 & 2 & 1 \end{bmatrix} , \quad \begin{bmatrix} 1 & 7 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

(4) Identity Matrix المصفوفة الاحادية (مصفوفة الوحدة)

Its a diagonal matrix whose elements on the main diagonal are all 1 and denoted by I_n

$$I_1 = [1] \quad , \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(5) Zero Matrix المصفوفة الصفرية

All the elements are zero

$$[0] \, , \, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} , \, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Mathematical Operations of Matrices

العمليات الرياضية للمصفوفات

Let :

$$A = [a_{jk}]_{m\times n} \quad \text{and} \quad B = [b_{jk}]_{m\times n} \qquad \cdots (2)$$

(1) Equality :

Two matrices A and B are equal if and only if A and B have the same number of rows and the same number of columns and corresponding elements are equal. That is $a_{jk} = b_{jk}$ for all occuring j and k.

(2) Addition of Matrices :

Is defined only for matrices having the same number of rows and columns and is defined as follows :

The sum of two $m\times n$ matrices $A = [a_{jk}]$ and $B = [b_{jk}]$ is the $m\times n$ matrix $C = [c_{jk}]$ with elements :

$$c_{jk} = a_{jk} + b_{jk} \qquad \begin{matrix} j = 1, 2, \cdots, m \\ k = 1, 2, \cdots, n \end{matrix} \qquad --- (3)$$

and written $C = A + B$.

Properties of Matrix Multiplication by Scalars

$$\left.\begin{array}{ll} (1) & q(A+B) = qA + qB \\ (2) & (c+k)\cdot A = cA + kA \\ (3) & c(kA) = ck(A) \\ (4) & 1\cdot A = A \end{array}\right\} \quad \cdots (5)$$

where q, c, and k are scalars

Transpose of A Matrix : مدور المصفوفة

The matrix of order $n\times m$ obtained by interchanging the rows and columns of an $m\times n$ matrix is called the transpose of A and is denoted A^T.

i.e.

The transpose of $A=[a_{jk}]=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

$$\text{is } A^T=[a_{kj}] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix} \quad \cdots (6)$$

For example:

The transpose of $A = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$ is $A^T = \begin{bmatrix} 3 & 6 \\ 4 & 7 \\ 5 & 8 \end{bmatrix}$

- A real square matrix $A=[a_{jk}]$ is said to be symmetric if it is equal to its transpose, $A^T=A$, that is $a_{kj}=a_{jk}$.

For example

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

then $A^T = A \Rightarrow A$ is symmetric matrix.

- A real square matrix $A = [a_{jk}]$ is said to be skew-symmetric if $A^T = -A$, that is

$$a_{kj} = -a_{jk}.$$

For example

$$A = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -5 \\ -1 & 5 & 0 \end{bmatrix}, \quad A^T = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 5 \\ 1 & -5 & 0 \end{bmatrix}$$

$$\therefore A^T = -A$$

- A real square matrix A may be written as the sum of a symmetric matrix R and a skew-symmetric matrix S, where

$$\left.\begin{aligned} R &= \tfrac{1}{2}(A + A^T) \\ S &= \tfrac{1}{2}(A - A^T) \\ R + S &= A \end{aligned}\right\} \quad \text{--- (7)}$$

Ex. The matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}$ is symmetric

and $A = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$ is skew-symmetric

while The matrix $A = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$ is neither symmetric nor skew-symmetric.

Properties of Determinants

(a) If two rows or columns are identical, the determinant is zero, for example

$$\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0 \quad , \quad \begin{vmatrix} -1 & -1 \\ -3 & -3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 & 3 \\ 0 & 4 & 3 \\ 2 & 1 & 3 \end{vmatrix} = 0 \quad , \quad \begin{vmatrix} 2 & 1 & 2 \\ 2 & 3 & 2 \\ 2 & 4 & 2 \end{vmatrix} = 0$$

(b) If all entries of a single row or column are zero, then the determinant is zero

for example:

$$\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} = 0 \quad , \quad \begin{vmatrix} 3 & 0 \\ 2 & 0 \end{vmatrix} = 0$$

(c) Interchanging any two rows or two columns alters the det. by a factor of -1

for example: $$\begin{vmatrix} 2 & 6 & 4 \\ 1 & 3 & 0 \\ -1 & 0 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 12$$

(d) $|A| = |A^T|$

(e) If A is a diagonal matrix or triangular matrix, the det A is the product of the elements on the main diagonal.

EX.

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \quad , \quad \begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} = -2 \quad , \quad \begin{vmatrix} 2 & 6 & -6 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{vmatrix} = (2)(3)(-1) = -6 .$$

Properties of Matrix Multiplication:

Assuming that A, B, C are confomable for the indicated sums and products. We have

$A(B+C) = AB + AC$ first distributive Law.
$(A+B)C = AC + BC$ second " " .
$A(BC) = (AB)C$ associative Law.

However,

$AB \neq BA$, generally
$AB = 0$ does not necessarily imply $A=0$ or $B=0$
$AB = AC$ " " " " $B=C$

Example:

Let $A = \begin{bmatrix} 4 & 1 & 2 & 1 \\ -3 & -1 & 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 4 & 1 \\ 1 & 1 & 6 & 0 \end{bmatrix}$

$C = \begin{bmatrix} x-1 & 3y \\ \frac{z}{2} & w-1 \\ q & p+1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 6 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$

Find

① $2A$, $-3B$, $\frac{2}{3}B$

② $A+B$, $A-B$, $2A-3B$

③ A^T, B^T, A^T+B^T, $2A^T$, $3B^T$

if $C=D$ find the values of x, y, z, w, q and p.

Solu.

① $2A = \begin{bmatrix} 8 & 2 & 4 & 2 \\ -6 & -2 & 0 & 6 \end{bmatrix}$, $-3B = \begin{bmatrix} -9 & 3 & -12 & -3 \\ -3 & -3 & -18 & 0 \end{bmatrix}$

$\frac{2}{3}B = \begin{bmatrix} 2 & -\frac{2}{3} & \frac{8}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & 4 & 0 \end{bmatrix}$.

② $$A+B = \begin{bmatrix} 4 & 1 & 2 & 1 \\ -3 & -1 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 4 & 1 \\ 1 & 1 & 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 0 & 6 & 2 \\ -2 & 0 & 6 & 3 \end{bmatrix}$$

$$A-B = \begin{bmatrix} 1 & 2 & -2 & 0 \\ -4 & -2 & -6 & 3 \end{bmatrix}$$

$$2A-3B = \begin{bmatrix} 8 & 2 & 4 & 2 \\ -6 & -2 & 0 & 6 \end{bmatrix} - \begin{bmatrix} 9 & -3 & 12 & 3 \\ 3 & 3 & 18 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -10 & -8 & -1 \\ -9 & -5 & -18 & 6 \end{bmatrix}$$

③ $$A^T = \begin{bmatrix} 4 & -3 \\ 1 & -1 \\ 2 & 0 \\ 1 & 3 \end{bmatrix}, \quad B^T = \begin{bmatrix} 3 & 1 \\ -1 & 1 \\ 4 & 6 \\ 1 & 0 \end{bmatrix}$$

$$2A^T = \begin{bmatrix} 8 & -6 \\ 2 & -2 \\ 4 & 0 \\ 2 & 6 \end{bmatrix} \qquad A^T+B^T = \begin{bmatrix} 7 & -2 \\ 0 & 0 \\ 6 & 6 \\ 2 & 3 \end{bmatrix}$$

$$3B^T = \begin{bmatrix} 9 & 3 \\ -3 & 3 \\ 12 & 18 \\ 3 & 0 \end{bmatrix}$$

④ $C=D$

$$\begin{bmatrix} x-1 & 3y \\ \frac{z}{2} & w-1 \\ q & p+1 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 2 & 0 \\ -1 & 2 \end{bmatrix} \Rightarrow$$

$x-1=1 \Rightarrow x=2$

$3y=6 \Rightarrow y=2$

$\frac{z}{2}=2 \Rightarrow z=4$

$w-1=0 \Rightarrow w=1$

$p+1=2 \Rightarrow p=1$

$q=-1 \Rightarrow q=-1$

Example

Let $A=\begin{bmatrix} 1 & -3 \\ 2 & 1 \\ 4 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 2 & -4 \\ 3 & 0 & -1 \end{bmatrix}$

$C=\begin{bmatrix} 2 & 3 \\ 7 & 1 \end{bmatrix}$

① Find AB ② Find BA ③ Is $AB=BA$

④ If its possible find CA and AC.

Solu.:

① $A \cdot B = \begin{bmatrix} 1 & -3 \\ 2 & 1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & -4 \\ 3 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -8 & 2 & -1 \\ 5 & 4 & -9 \\ 4 & 8 & -16 \end{bmatrix}$

② $B \cdot A = \begin{bmatrix} 1 & 2 & -4 \\ 3 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & 1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} -11 & -1 \\ -1 & -9 \end{bmatrix}$

③ $AB \neq BA$

④ $CA \longrightarrow C_{2\times 2} \quad A_{3\times 2}$, then CA is not defined

$2 \neq 3$

$$AC = \begin{bmatrix} 1 & -3 \\ 2 & 1 \\ 4 & 0 \end{bmatrix}_{3\times 2} \cdot \begin{bmatrix} 2 & 3 \\ 7 & 1 \end{bmatrix}_{2\times 2} = \begin{bmatrix} -19 & 0 \\ 11 & 7 \\ 8 & 12 \end{bmatrix}$$

Submatrix

Any matrix obtained by omitting some rows and columns from a given $m \times n$ matrix A is called a submatrix of A.

Example

Submatrices of a matrix

The matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

Contains: three 2×2 submatrices, namely

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}, \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

and two 1×3 submatrices (the two row vectors)

$$[a_{11} \quad a_{12} \quad a_{13}], \; [a_{21} \quad a_{22} \quad a_{23}]$$

It also contains

three 2×1 submatrices (the column vectors)

$$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}, \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

and six 1×2 submatrices:

$$[a_{11} \quad a_{12}], \; [a_{11} \quad a_{13}], \; [a_{12} \quad a_{13}]$$
$$[a_{21} \quad a_{22}], \; [a_{21} \quad a_{23}], \; [a_{22} \quad a_{23}]$$

and six 1×1 submatrices:

$$[a_{11}], [a_{12}], [a_{13}], [a_{21}], [a_{22}], [a_{23}].$$

Minors and Cofactor :

Each element a_{jk} of a determinant A has minors M_{jk} which is a determinant obtained by deleting the j^{th} row and k^{th} column.

EX. The third order determinant $\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

The minor of a_{12} is the second order determinant $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31}$

The Cofactor A_{jk} is defined as :

$$A_{jk} = (-1)^{j+k} M_{jk} \qquad \text{--- (10)}$$

Example :

The cofactor of a_{12} is

$$A_{12} = (-1)^{1+2} M_{12}$$

$$= - \begin{vmatrix} a_{21} & a_{33} \\ a_{31} & a_{33} \end{vmatrix}$$

OR The sign array may be written as

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

now, the determinant

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

may be evaluated by det $A = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$ and may be performed in five order ways using the elements of the other two rows and three Columns.

Example:

Determine the minor M_{23} of $\begin{bmatrix} 1 & 2 & -1 \\ 4 & 3 & -4 \\ -2 & 0 & -3 \end{bmatrix}$

Solu.

First delete the 2^{nd} row and 3^{rd} column which yeilds the matrix $\begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}$

then M_{23} is the determinant of this matrix

$$M_{23} = \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} = (1)(0) - (-2)(2) = 4.$$

Example

Evaluate $\begin{vmatrix} -3 & 2 & 1 \\ -5 & 0 & 6 \\ -2 & -1 & 3 \end{vmatrix}$

Solu.

choose the row or column where there is a number 0.

$$= -(-5)\begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} + 0 - 6\begin{vmatrix} -3 & 2 \\ -2 & -1 \end{vmatrix}$$

$$= 5[6-(-1)] - 6[(-3)(-1) - (-2)(2)]$$

$$= 35 - 42 = -7.$$

f. If each element of some rows or columns of a matrix A is a multiplied by a constant k then the determinant is multiplied by k.

Ex.

$$\begin{vmatrix} 3 & 3 \\ 5 & -1 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} = (3)\left[(1)(-1) - (5)(1)\right] = 3(-1-5) = -18$$

g. If we interchange the i^{th} and the j^{th} row of the matrix A the the value of the determinant is change by a factor of -1

$$54 = \begin{vmatrix} 2 & 3 & 1 \\ 4 & -2 & 0 \\ 5 & 7 & -1 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{vmatrix} 5 & 7 & -1 \\ 4 & -2 & 0 \\ 2 & 3 & 1 \end{vmatrix} = -54$$

h. If we add k times the j^{th} row to the i^{th} row and put it in place of the i^{th} row, then the determinant still same.

$$-12 = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \\ 3 & 2 & 5 \end{vmatrix} \xrightarrow[R_3 \leftrightarrow R_2 + R_3]{R_1 \leftrightarrow -R_2 + R_1} \begin{vmatrix} -3 & 2 & -2 \\ 4 & 0 & 1 \\ 7 & 2 & 6 \end{vmatrix} = -12$$

Ex. for the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

$a_{11} = \begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} = 6$, $a_{12} = \begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} = -2$

$a_{13} = \begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} = -3$, $a_{21} = 1$, $a_{22} = -5$

$a_{23} = 3$, $a_{31} = -5$, $a_{32} = 4$, $a_{33} = -1$

and $\text{adj } A = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix} \Rightarrow \text{adj } A \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \end{bmatrix}$

EX.(1) Addition of Matrices:

$$A = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

then

$$A + B = \begin{bmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{bmatrix}.$$

Properties of Matrix addition:

1. $A + B = B + A$
2. $A + 0 = A$ --- (4)
3. $A + (-A) = 0$

where $-A = [-a_{jk}]$ is the $m \times n$ matrix obtained by multiplying every element of A by (-1) and is called the negative of A.

Multiplication of Matrices by Scalars (numbers).

The product of an $m \times n$ matrix A by a scalar $\mathfrak{z}$ is denoted by $\mathfrak{z}A$ or $A\mathfrak{z}$ and is the $m \times n$ matrix obtained by multiplying every element of A by $\mathfrak{z}$.

$$\mathfrak{z}A = A\mathfrak{z} = \begin{bmatrix} \mathfrak{z}a_{11} & \mathfrak{z}a_{12} & \cdots & \mathfrak{z}a_{1n} \\ \mathfrak{z}a_{21} & \mathfrak{z}a_{22} & \cdots & \mathfrak{z}a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \mathfrak{z}a_{m1} & \mathfrak{z}a_{m2} & \cdots & \mathfrak{z}a_{mn} \end{bmatrix}$$

EX. If $A = \begin{bmatrix} 2.7 & -1.8 \\ 0.9 & 3.6 \end{bmatrix}$ Calculate $2A$ and $\frac{10}{9}A$

Solu.

$$2A = \begin{bmatrix} 5.4 & -3.6 \\ 1.8 & 7.2 \end{bmatrix} \text{ and } \frac{10}{9}A = \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}$$

Using Cramer's Rule 3 X 3 Linear System

$$2x + y - z = 1 \qquad\qquad a_1 x + b_1 y + c_1 z = d_1$$
$$3x + 2y + 2z = 13 \implies a_2 x + b_2 y + c_2 z = d_2$$
$$4x - 2y + 3z = 9 \qquad\qquad a_3 x + b_3 y + c_3 z = d_3$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 2 \\ 4 & -2 & 3 \end{bmatrix} \implies \det(A) = 2\begin{bmatrix} 2 & 2 \\ -2 & 3 \end{bmatrix} - 1\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} - 1\begin{bmatrix} 3 & 2 \\ 4 & -2 \end{bmatrix}$$

find $|A|$ $\det(A)$

hint $\det(A)$ or $|A|$

or

$$|A| = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$2\left[(2(3) - (2)(-2)\right] - 1\left[9 - 8\right] - \left[-6 - 8\right]$$
$$2(6 + 4) - 1(1) - (-14)$$
$$20 - 1 + 14 = 33$$

$$\boxed{D = 33}$$

$$Dx = \begin{bmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 13 & 2 & 2 \\ 9 & -2 & 3 \end{bmatrix} \implies Dx = 1\begin{bmatrix} 2 & 2 \\ -2 & 3 \end{bmatrix} - 1\begin{bmatrix} 13 & 2 \\ 9 & 3 \end{bmatrix} + (-1)\begin{bmatrix} 13 & 2 \\ 9 & -2 \end{bmatrix}$$

$$Dx = 1(6 - (-4)) - 1(39 - 18) - 1(-26 - 18)$$
$$Dx = 10 - 21 + 44 = 33$$
$$\boxed{Dx = 33}$$

$$Dy = \begin{bmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 13 & 2 \\ 4 & 9 & 3 \end{bmatrix} \implies Dy = 2\begin{bmatrix} 13 & 2 \\ 9 & 3 \end{bmatrix} - 1\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} + (-1)\begin{bmatrix} 3 & 13 \\ 4 & 9 \end{bmatrix}$$

$$\implies Dy = 2(39 - 18) - (9 - 8) - 1(27 - 52)$$
$$Dy = 2(21) - (1) - (-25)$$
$$Dy = 42 - 1 + 25 = 66$$

$$\boxed{Dy = 66}$$

Inverse Matrix

if A is a square matrix of order n, then A has an inverse if and only if there is a square matrix B of order n such that $AB = I_n$.

B is called inverse of A and written A^{-1}

$\therefore \; AA^{-1} = I_n$ and $A^{-1}A = I_n$ --- (12)

A square matrix has an inverse if its determinant $\neq 0$

To find the inverse of a matrix whose (determinant is not zero).

(a) Construct the matrix ~~whose~~ of cofactor of A

$$\text{cof}(A) = C_{ij}$$

(b) Construct the transposed matrix of cofactors called (the adjoint of A)

$$\text{adj}(A) = (\text{cof}\,A)^T$$

(c) Then $A^{-1} = \dfrac{\text{adj}(A)}{\det A}$

(d) To check your answer $A \cdot A^{-1} = I$

Example: Find the inverse of the matrix $A = \begin{bmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \\ 3 & -1 & -1 \end{bmatrix}$

Solu.

$$C_{11} = (-1)^{1+1} A_{11} = \begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} = -2+3 = 1$$

$$C_{12} = (-1)^{1+2} A_{12} = -\begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix} = 10$$

$$C_{13} = (-1)^{1+3} A_{13} = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -1-6 = -7$$

$$C_{21} = (-1)^{2+1} A_{21} = \begin{vmatrix} 3 & -4 \\ -1 & -1 \end{vmatrix} = -(-3-4) = 7$$

$$C_{22} = (-1)^{2+2} A_{22} = \begin{vmatrix} 2 & -4 \\ 3 & -1 \end{vmatrix} = -2+12 = 10$$

$$C_{23} = (-1)^{2+3} A_{23} = \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -(-2-9) = 11$$

$$C_{31} = (-1)^{3+1} A_{31} = \begin{vmatrix} 3 & -4 \\ 2 & 3 \end{vmatrix} = 9+8 = 17$$

$$C_{32} = (-1)^{3+2} A_{32} = -\begin{vmatrix} 2 & -4 \\ 1 & 3 \end{vmatrix} = -(6+4) = -10$$

$$C_{33} = (-1)^{3+3} A_{33} = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 4-3 = 1$$

$$Cof(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 1 & 10 & -7 \\ 7 & 10 & 11 \\ 17 & -10 & 1 \end{bmatrix}$$

$$\operatorname{adj} A = (Cof\, A)^T = \begin{bmatrix} 1 & 7 & 17 \\ 10 & 10 & -10 \\ -7 & 11 & 1 \end{bmatrix}$$

$$\det(A) = |A| = \begin{vmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \\ 3 & -1 & -1 \end{vmatrix} = 60$$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A) = \frac{1}{60} \begin{bmatrix} 1 & 7 & 17 \\ 10 & 10 & -10 \\ -7 & 11 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{60} & \frac{7}{60} & \frac{17}{60} \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \\ \frac{-7}{60} & \frac{11}{60} & \frac{1}{60} \end{bmatrix}$$

Linear Equations المعادلات الخطية

Consider a system of Linear equations in three unknowns x, y, z

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}\right\} \quad \text{--- (13)}$$

In matrix notation the system of Linear eq.s may be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{--- (14)}$$

The above theorem called <u>Grammer's Rule</u>.

To solve the above theorem we put :

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \quad D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

if $D \neq 0$, then the system has a unique solution which is

$$x = \frac{D_1}{D}, \quad y = \frac{D_2}{D}, \quad z = \frac{D_3}{D}.$$

EX. : Use Grammer's Rule to solve

$$5x - 2y = -1$$
$$2x + 3y = 3$$

Solu.

$$\begin{bmatrix} 5 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$D = \begin{vmatrix} 5 & -2 \\ 2 & 3 \end{vmatrix} = 15 + 4 = 19$$

$$D_1 = \begin{vmatrix} -1 & -2 \\ 3 & 3 \end{vmatrix} = -3 + 6 = 3 \quad ; \quad D_2 = \begin{vmatrix} 5 & -1 \\ 2 & 3 \end{vmatrix} = 15 + 2 = 17$$

$$\therefore X = \frac{D_1}{D} = \frac{3}{19} \quad , \quad y = \frac{D_2}{D} = \frac{17}{19} .$$

EX. Use Grammer's Rule to solve

$$x + \quad + 2z = 6$$
$$-3x + 4y + 6z = 30$$
$$-x - 2y + 3z = 8$$

Solu:

$$D = \begin{vmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix} = +11 \quad , \quad D_1 = \begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{vmatrix} = -10$$

$$D_2 = \begin{vmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{vmatrix} = 18 \quad , \quad D_3 = \begin{vmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{vmatrix} = 38$$

$$X = \frac{D_1}{D} = \frac{-10}{11} \quad , \quad y = \frac{D_2}{D} = \frac{18}{11} \quad , \quad z = \frac{D_3}{D} = \frac{38}{11}$$

Solution of a System of Linear Equations by: (Inverse Matrix Method)

Consider the system of n Linear equations in n unknowns x_i,

$$AX = B \qquad (15)$$

where A is an $n \times n$ matrix $[a_{jk}]$, X is the column matrix $[x_i]$ and B is the column matrix $[b_i]$. If A is non-singular ($\det A \neq 0$), then A^{-1} exists and equation (15) may be written

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

and $$X = A^{-1}B \qquad (16)$$

Equation (16) is then the formal solution of equation (15).

The solution (16) may be expressed in terms of determinants and cofactors. As illustration consider the case $n=3$, then

$$X = \frac{1}{\det A}(\operatorname{adj} A)B = \frac{1}{\det A}\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$X = \frac{1}{\det A}\begin{bmatrix} b_1 a_{11} + b_2 a_{21} + b_3 a_{31} \\ b_1 a_{12} + b_2 a_{22} + b_3 a_{32} \\ b_1 a_{13} + b_2 a_{23} + b_3 a_{33} \end{bmatrix} \qquad (17)$$

<u>Example</u> Solve the System by (Inverse Matrix Method):

$$2x_1 + x_2 - x_3 = 4$$

$$x_1 - 2x_2 + x_3 = 10$$

$$-3x_1 \qquad - 2x_3 = 9$$

<u>Solu.</u>

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ -3 & 0 & -2 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ -10 \\ 9 \end{bmatrix}$$

Since $\det A = (-3)(-1) + (-2)(-5) = 13$;

$$A^{-1} = \frac{1}{13}\begin{bmatrix} 4 & 2 & -1 \\ -1 & -7 & -3 \\ -6 & -3 & -5 \end{bmatrix}$$

and $$X = \frac{1}{3}\begin{bmatrix} 4 & 2 & -1 \\ -1 & -7 & -8 \\ -6 & -3 & -5 \end{bmatrix}\begin{bmatrix} 4 \\ -10 \\ 9 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} -13 \\ 39 \\ -39 \end{bmatrix}$$

Hence $X_1 = -1$, $X_2 = 3$; $X_3 = -3$

Exercises

① Solve by the Grammer's Rule

(a)
$$2X_1 - 5X_2 = -2$$
$$4X_1 + 6X_2 = 1$$

(b)
$$x + 2y + z = 3$$
$$2x + y - z = 0$$
$$x - y + z = 0$$

② Find the inverse matrix of A

(a) when $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

(b) when $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

③ If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & -2 \\ 1 & -5 \\ 4 & 3 \end{bmatrix}$ Find $D = \begin{bmatrix} p & q \\ r & s \\ t & u \end{bmatrix}$

such that $A + B - D = 0$.

$$Dz = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 13 \\ 4 & -2 & 9 \end{bmatrix} \Rightarrow Dz = \begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix}$$

$$Dz = 2\begin{vmatrix} 2 & 13 \\ -2 & 9 \end{vmatrix} - 1\begin{vmatrix} 3 & 13 \\ 4 & 9 \end{vmatrix} + 1\begin{vmatrix} 3 & 2 \\ 4 & -2 \end{vmatrix}$$

$$Dz = 2(18-26) - 1(27-52) + 1(-6-8)$$

$$Dz = 2(44) - 1(-25) + (-14) = 99$$

$$\boxed{Dz = 99}$$

$$x = \frac{Dx}{D} = \frac{33}{33} = 1$$

$$y = \frac{Dy}{D} = \frac{66}{33} = 2$$

$$z = \frac{Dz}{D} = \frac{99}{33} = 3$$

$$\therefore \boxed{x = 1}, \boxed{y = 2}, \boxed{z = 3}$$