Engineering Mathematics

3rd - level

Civil Engineering department

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Subjects :

- → Ordinary differential equation
- → Linear differential equation (Second order)
- → Application of second order differential equation.
- → High order differential equation
- → Simultaneous differential equation
- → Application of Simultanoous differential equation.
- → Foruier Series
- → Application of foruier Series
- → partial differential equation.

الموضوع المعادلات التفاضلية

ملاحظة :- المعادلات التفاضلية هي معادلات التي تحتوي على متغيرين x, y احدهما معتمد و الثاني غير معتمد

$$\frac{dy}{dx} + x = \sin x$$

(y يعتبر الاساسي , x تعتبر غير معتمد)

ملاحظة :- اذا لم تحل المعادلة نقوم بقلبها (اي قلب كافة الحدود).

ويجب ان تحتوي المعادلة التفاضلية على مشتقة واحدة على الأقل.

order
* رتبة :- هي اعلى مشتقة في المعادلة

* درجة :- اكبر أس موجود على اعلى مشتقة بشرط التخلص من كافة الاسس الكسرية
degree

ex) $10 - (y'') = (y')^{3/2}$ نربع طرفين للتخلص من أس المتكون من كسر $\underline{\underline{3/2}}$

$$100 - 20y'' + (y'')^{2} = (y')^{3}$$

رتبة order $= 2^{nd}$

درجة degree $= \cancel{1^{st}}$ 2nd

الموضوع هنالك نوع من الانواع لا
يحل بالفصل بين المتغيرات.

2

ملاحظة! المعادلة التفاضلية التي يتساوى فيها درجة بسط مع مقام تسمى متجانسة

وفي حالة وجود ثوابت يجب ان نتخلص منها لكي نحولها الى معادلة تفاضلية متجانسة.

المعادلة التفاضلية قد تحتوي على مستقيمات متقاطعات او متوازيان ويمكن معرفة ذلك من خلال الاشارات تبع المتغيرات.

$$\frac{dy}{dx} = \frac{x+y+3}{x-y-3}$$ متقاطعات

$$\frac{dy}{dx} = \frac{2x+y+5}{2x+y-3}$$ متوازيان.

هنالك نوع يسمى معادلة تفاضلية تامة

= = = = = خطية

شروطها $$\frac{dy}{dx} + xy = \sin x$$

1 يجب ان يكون طرف اليمين خالي من y و 2 الـ y في الطرف الايسر مرفوعة لأس واحد

وفي حالة حدوث خلل بأحد الشروط تعتبر على غير خطية.

Ordinary Differential Equation O.D.E

المعادلات التفاضلية الاعتيادية :- هي المعادلات التي تحتوي على متغير مستقل وليكن x

Independent Variable

ومتغير تابع وليكن $y(x)$ يُسمى dependent Variable

وواحد او اكثر من المشتقات التفاضلية $y', y'', \ldots$

اي تكون المعادلة التفاضلية بصيغة :-

$F(x, y, y', y'' \ldots]$

وتسمى هذه معادلة تفاضلية إعتيادية. أما اذا كان عدد المتغيرات المستقلة اكثر من واحد وليكن (x, y, t) وكان $z(x, y, t)$ متغير تابع قابل للاشتقاق لكل من x, y, t --- جزئياً. تسمى المعادلة التي تشتمل على المتغيرات المستقلة والمتغير التابع و مشتقاته الجزئية بالمعادلة التفاضلية الجزئية وتكون Partial differential equation

على الشكل التالي: $G(x, y, t, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}) = 0$

مثال ذلك

$$y' + xy = x^2 \quad \dots (1)$$

$$y''' + 2y'' - 5y = \sin x \quad \dots (2)$$

$$\frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} = x \quad (3)$$

O.D.E

حيث معادلة (1) و (2) معادلة تفاضلية إعتيادية

معادلة (3) معادلة تفاضلية جزئية.

رتبة المعادلة التفاضلية Order of differential Equation

هي رتبة أعلى معامل تفاضلي (مشتقة) في المعادلة التفاضلية

درجة المعادلة التفاضلية

هي درجة (قوة، اس) اعلى معامل تفاضلي في المعادلة بشرط ان تكون جميع المعاملات التفاضلية خالية من ~~الاسس الكسرية~~

بالرجوع الى معادلات ١ & ٢ في معادلة ٢

مرتبة ١ درجة ١

معادلة ٢ مرتبة 3 درجة ١ .

مثال/ بتربيع طرفين $y'' = (5 - 2y')^{3/2}$

$(y'')^2 = (5 - 2y')^3$

مرتبة ٢ درجة ٢

- تكوين المعادلة التفاضلية Formation of D.E

١) طريقة حذف الثوابت : لتكن $F(x, y, c_1, c_2, \ldots) = 0$ ②

هو الحل العام لمعادلة تفاضلية من المرتبة n حيث

ان الحل العام يعتمد على n من الثوابت الاختيارية

حيث $c_1, c_2, \ldots c_n$ ثوابت اختيارية . الحصول

على المعادلة التفاضلية للحل المعطى نجري n من

المشتقات للمعادلة ① حيث سنشتق حسب

عدد الثوابت .

Chapter 1

Ordinary Differential Equations of the First order

رموز المشتقة $\frac{dy}{dx} = D_x = y' = \nabla y$

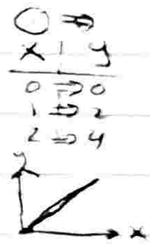
Differential Equation :- المعادلة التفاضلية

هي المعادلة التي تحتوي على الأقل على مشتقة واحدة

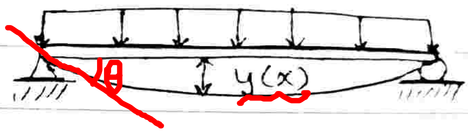
$$\frac{dy}{dx} = f(x)$$

$y = f(x)$

مثال $y = 2x$

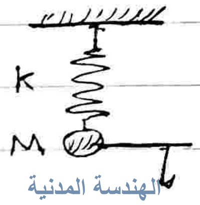


$$y' = \frac{M(x)}{EI(x)} \quad \text{bending}$$



Independent variable — المتغير المستقل

dependent variable — المتغير التابع



t ; independent variable

y : dependent variable

الحل : يمثل علاقة بين المتغير التابع والمستقل خالية من المشتقات وتحقق المعادلة التفاضلية

Differential Equation

Ordinary Diff. Equation

1- تحتوي مشتقات اعتيادية فقط وتظهر هذه المعادلات عندما تكون هناك one independent variable

Partial Diff. Equation.

تحوي مشتقات للدرجة

y = deflection

y' = Slope

y'' = Bending Moment

y''' = Shear

y'''' = Load

Partial differential equation :-

dependent variable $u(x, y, z, t) \Longrightarrow$ One partial differential equation

لان u واحد

تكون المشتقة partial differential equation عندما يكون عدد المتغيرات المستقلة اكثر من واحد

Order مرتبة (اعلى مشتقة)

degree درجة (اس اعلى مشتقة)

example :- $x^2 \frac{d^2y}{dx^2} + \frac{dy}{dx} + (x^2 - 4) = f(x)$

Second order ordinary differential equation

معادلة تفاضلية إعتيادية من المرتبة الثانية

$$\frac{\partial^4 u}{\partial x^4} + \frac{2\partial}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0$$

معادلة تفاضلية جزئية من المرتبة الرابعة

Degree
- Linear Differential equation معادلة تفاضلية خطية
- Non linear Differential equation معادلة تفاضلية لا خطية

مثلا $y = x^2$

المعادلة التفاضلية الخطية :- هي تلك المعادلة التي كل حد من حدودها هو خطي بالنسبة للمتغير التابع ومشتقاته ، اي يجب ان تكون (y) كمتغير تابع ومشتقاتها y ، y' ، ..... ذات مساوي الى 1 ولوحدها اي ان لا تكون مثل $\sin y$ ، e^y ، $\ln y$ ، $\frac{1}{y}$ الخ ..... وهي سهلة الحل

المعادلة التفاضلية اللا خطية :- وهي معادلة صعبة الحل ولكن المهندس يحاول ان يحل لها ما يسمى بـ Linearization مثل :-

$$\frac{y''}{[1+y'^2]^{3/2}} = \frac{M(x)}{EI(x)} \Longleftarrow \text{ينتج} \quad y'' = \frac{M(x)}{EI(x)}$$

Examples :-

1- $\overset{lin}{y''} + 4x\overset{lin}{y'} + 2\overset{lin}{y} = \sin x$ } معادلة تفاضلية خطية من المرتبة الثانية

2- $y'' + 4\overset{Non\ lin}{y y'} + 2y = \sin x$ } معادلة تفاضلية لا خطية (بسبب ضرب y مع y') من المرتبة الثانية

3- $y'' + \sin \overset{Non\ lin}{(y)} = 0$ } معادلة تفاضلية لاخطية (بسبب وجود (sin) من المرتبة الثانية

حل المعادلة التفاضلية تحتوي على ثوابت اختيارية

$$\frac{dy}{dx} = f(x) \quad \text{(بالتكامل)}$$

$$y(x) = \int f(x)\,dx + C$$

Essential Arbitrary Constants

س: هل ان جميع الثوابت الاختيارية هي ضرورية ؟

ج ← تعتبر الثوابت الاختيارية في الحل ضرورية اذا لم يكن بالإمكان تقليص عددها بأي إعادة ترتيب

e.g → $y(x) = a\cos^2 x + b\sin^2 x + C\cos 2x$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$y(x) = a\cos^2 x + b\sin^2 x + C[\cos^2 x - \sin^2 x]$$

$$y(x) = (\underbrace{a+C}_{D})\cos^2 x + (\underbrace{b-C}_{E})\sin^2 x$$

$$y(x) = D\cos^2 x + E\sin^2 x$$

هل ان الثوابت a, b, c ضرورية

* بصورة عامة: عدد الثوابت الاختيارية في الحل لا يجوز أن يزيد عن رتبة المعادلة التفاضلية

e.g

1- $\left|\frac{dy}{dx}\right| + |y| = 0$

∴ الثوابت = صفر

$$\left|\frac{dy}{dx}\right| = -|y| \Rightarrow y = 0$$

2- $\left|\frac{dy}{dx}\right| + 1 = 0$ هذه المعادلة لا تملك حل على الاطلاق

Solutions of Differential Equation

1- General Solution :-

الحل العام :-

هو ذلك الحل الذي يحتوي على الأقل على ثابت إختياري واحد

eg $y'' = \frac{M(x)}{EI(x)}$

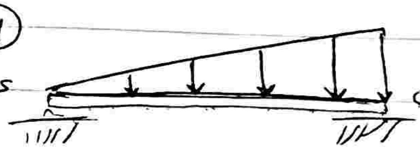
هذا الحل يصلح لجميع حالات الجسور

2. Particulate Solution :-

الحل الخاص

هو ذلك الحل الناتج عن الحل العام بعد اعطاء قيم خاصة للثوابت الاختيارية

(4) ---- Conditions



يحتاج الى حل خاص

3- Singular Solutions :-

الحل الشاذ

هو ذلك الحل الذي لا يمكن الحصول عليه من الحل العام لإعطاء قيم خاصة للثوابت الاختيارية ولهذا سمي بالحل الشاذ

4- Complete solution :-

هو ذلك الحل الذي بالإمكان الحصول منه على جميع حلول المعادلة التفاضلية.

مثال 1 :- اثبت ان $y = ae^{-x} + be^{2x}$ هو حل المعادلة التفاضلية التالية لجميع قيم الثوابت a, b :-

$$\boxed{y'' - y' - 2y = 0}$$

$$\left.\begin{aligned} y &= ae^{-x} + be^{2x} \\ y' &= -ae^{-x} + 2be^{2x} \\ y'' &= ae^{-x} + 4be^{2x} \end{aligned}\right\}$$

نشتق ونعوض بالدالة

$$[ae^{-x} + 4be^{2x}] - [-ae^{-x} + 2be^{2x}] - 2[ae^{-x} + be^{2x}] \overset{?}{=} 0$$

$$[\cancel{e^{-x}} + \cancel{e^{-x}} - \cancel{2e^{-x}}]a + [4be^{2x} - 2be^{2x} - 2be^{2x}]b \overset{?}{=} 0$$

$$0 \cdot a + 0 \cdot b = 0$$

مثال (2) ـ إثبت ان $y=ae^{-x}+be^{2x}$ لا يمكن ان يكون حلاً للمعادلة التفاضلية الآتية لجميع قيم الثوابت a, b

$$\boxed{y y'' - (y')^2 = 0}$$

$$y' = -ae^{-x} + 2be^{2x}$$

$$y'' = ae^{-x} + 4be^{2x}$$

$$(ae^{-x}+be^{2x})(ae^{-x}+4be^{2x}) - (-ae^{-x}+2be^{2x})^2$$

$$= \cancel{a^2e^{-2x}} - 4abe^{x} + abe^{x} + \cancel{4b^2e^{4x}} - (\cancel{a^2e^{-2x}} - 4abe^{x} + \cancel{4b^2e^{4x}})$$

تجمع

$$= 9abe^{x} \neq 0 \quad \text{except when } a=0 \text{ or } b=0$$

لاحظ هنا في المثال السابق (1) كان الناتج = صفر فيما اذا كانت a = صفر (أو) b = صفر

وعندما لا يساوي كل منهما صفر المهم ان حاصل مجموعهما يساوي صفر في المثال الثاني فقد اشترط الحل كون a = صفر أو b = صفر ان معادلة المثال الاول معادلة خطية بينما معادلة المثال الثاني معادلة لا خطية ومن هنا نستنتج بصورة عامة :- يصح مبدأ التراكب super position في المعادلات التفاضلية الخطية ولا يصح هذا المبدأ في المعادلات التفاضلية اللاخطية .

مثال ٣ ـ اذا كانت a, b ثوابت اختيارية , جد معادلة تفاضلية من المرتبة الثانية التي تملك الصيغة التالية كحل عام لها :-

$$y = ae^{x} + b\cos x \quad \text{----} \; (1)$$

$$y' = ae^{x} - b\sin x \quad \text{----} \; (2)$$

$$y'' = ae^{x} - b\cos x \quad \text{----} \; (3)$$

$$(1)+(3) \Rightarrow y + y'' = 2ae^{x} \Rightarrow \boxed{a = \frac{y+y''}{2e^{x}}}$$

$$(1)-(3) \Rightarrow y - y'' = 2b\cos x \Rightarrow \boxed{b = \frac{y-y''}{2\cos x}}$$

Sub. in (2)
$$y' = \frac{y+y''}{2\cancel{e^{x}}}\cancel{e^{x}} - \frac{y-y''}{2\cos x}\sin x$$

لاحظ اذا عوضنا a و b في احدى المعادلتين ينتج صفر = صفر

$$y' = \frac{y+y''}{2} - \frac{y-y''}{2}\tan x$$

$\therefore (1+\tan x)y'' - 2y' + (1-\tan x)y = 0$

هذه المعادلة - هي المعادلة التفاضلية الوحيدة من المرتبة الثانية التي تملك الحل أعلاه و لكنها ليس المعادلة الوحيدة لأي رتبة

From ③ :-

$y''' = ae^x + b\sin x$

$y^{IV} = ae^x + b\cos x \Rightarrow \boxed{y^{IV} = y}$ محادلة تفاضلية لنفس الحل

Exercises :- ① $y = ae^{3x} + bxe^{3x}$

sol. $y'' - 6y' + 9y = 0$, $y = ae^{3x} + bxe^{3x}$

$y' = 3ae^{3x} + b(3xe^{3x} + e^{3x})$

$y'' = 9ae^{3x} + b(9xe^{3x} + 3e^{3x} + 3e^{3x})$

$9ae^{3x} + 9bxe^{3x} + 6be^{3x} - 18ae^{3x} - 8be^{3x} - 6be^{3x} + 9ae^{3x} + 9bxe^{3x} \stackrel{?}{=} 0$

②

sol. $y'' + (y')^2 + 1 = 0$

$\left.\begin{aligned} y &= |\cos(x-a)| + b \\ y' &= -\tan(x-a) \\ y'' &= -\sec^2(x-a) \end{aligned}\right\}$ given

$-\sec^2(x-a) + (-\tan(x-a))^2 + 1 \stackrel{?}{=} 0$

$\sec^2(x-a) - \tan^2(x-a) = 1$ ok

③ $x^2y'' - 2xy' + 2y = 0$

لاحظ هنا في هذه الاسئلة يقول السائل جد المعادلة التفاضلية لأقل مرتبة وأقل مرتبة تعني مساوية الى عدد الثوابت في المعادلة a, b

$\rightarrow y = 2ax + bx^2$ ----- ①

$y' = 2a + 2bx$ ----- ②

$y'' = 2b$ ----- ③

نشتق مرتين لوجود ثابتين

from ③ $b = \frac{y''}{2}$

sub. in ② $y' = 2a + y''x \Rightarrow a = \frac{y' - 2y''x}{2}$

sub. in ① $y = xy' - x^2y'' + \frac{x^2y''}{2}$

$$x^2y'' - 2xy' + 2y = 0$$

④ $y = ae^{-x} + be^{x} + ce^{2x} \Longrightarrow$ تشتق للاث مرات

Sol.

$(*) \Rightarrow y = ae^{-x} + be^{x} + ce^{2x}$ ------ ①

$y' = -ae^{-x} + be^{x} + 2ce^{2x}$ ------ ②

$y'' = ae^{-x} + be^{x} + 4ce^{2x}$ ------ ③

$y''' = -ae^{-x} + be^{x} + 8ce^{2x}$ ------ ④

① - ② $\Longrightarrow y - y' = 2ae^{-x} - ce^{2x}$ ------ ⑤

② - ③ $\Longrightarrow y' - y'' = -2ae^{-x} - 2ce^{2x}$ ------ ⑥

⑤ + ⑥ $\Longrightarrow y - y'' = -3 \Longrightarrow \boxed{c = -\frac{y - y''}{3e^{2x}}}$

Sub. in ⑤ $\Rightarrow y - y' = 2ae^{-x} + \frac{2(y - y'')}{3} \Rightarrow \boxed{a = \frac{y - y' + \frac{2}{3}y - \frac{2}{3}y''}{2e^{-x}}}$

Sub. in ① $\Rightarrow y = \frac{y - y' + \frac{2}{3}y - \frac{2}{3}y''}{2} + be^{x} + \frac{y - y''}{3}$

$$b = \frac{y - \frac{y}{2} + \frac{y'}{2} - \frac{1}{3}y + \frac{1}{3}y'' - \frac{y}{3} + \frac{y''}{3}}{e^{x}} = \frac{\frac{1}{6}y + \frac{1}{2}y' + \frac{2}{3}y''}{e^{x}}$$

Sub. a, b, c, in ④:

$$y''' = -\frac{y - y' + \frac{2}{3}y - \frac{2}{3}y''}{2} - \frac{1}{6}y + \frac{1}{2}y' + \frac{2}{3}y'' - \frac{8(y - y'')}{3}$$

Linear Ordinary Diff. equations with Constant Coefficients

→ The Homogenous Linear Differential equations with Constant Coefficients

" المعادلات الخطية المتجانسة ذات المعاملات الثابتة *

$$ay'' + by' + cy = f(x) \leftarrow R(x)$$

$$\boxed{ay'' + by' + cy = 0} \quad a, b, c = \text{constants}$$ (الصيغة العامة)

Operators المؤثرات

$$D = \frac{d}{dx}$$

$$Dy = \frac{dy}{dx}$$

$$D^2y = D(Dy) = \frac{d^2y}{dx^2}$$

$$D^3y = \frac{d^3y}{dx^2}$$

$$(aD^2 + bD + c)y = aD^2y + bDy + cy$$

Example :-

$$(3D^2 - 10D - 8)x^2 = 3D^2x^2 - 10Dx^2 - 8x^2$$

$$= 6 - 20x - 8x^2$$

option 1 $(3D+2)(D-4)x^2 = (3D+2)(2x - 4x^2)$ $\quad Dx^2 - 4x^2$ — يبقى ثابت في (x) محله .

$3Dx - 12x^2D$

$$= 6 - 24x + 4x - 8x^2$$

$$= 6 - 20x - 8x^2$$

option 2 $(D-4)(3D+2)x^2 = (D-4)(6x + 2x^2)$

$$= 6 + 4x - 24x - 8x^2$$

$$= 6 - 20x - 8x^2$$

$$\boxed{(aD^2 + bD + c)y = 0}$$

$$D^2y \Rightarrow \frac{d^2y}{dx^2} \neq Dy = \frac{dy}{dx} \text{ or } \frac{dx}{dy}$$

c2e^m2x

$$m_1 \neq m_2 \Longrightarrow y_h = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$m_1 = m_2 \Longrightarrow y_h = C_1 e^{m_1 x} + C_2 x e^{m_2 x}$$

m_1 or $m_2 = \alpha + \beta i$ اعداد خيالية

مثلاً $\sqrt{(-1)}$

$$y_h = C_1 e^{(\alpha+\beta)x} + C_2 e^{(\alpha+\beta)x}$$

or $y_h =$

$$y_g = y_h + y_p$$

ماقبل المساوي

مابعد المساويه

(x-2)(x+3)=0
x=2, m1
x=-3, m2

$$\{ay'' + by' + cy = 0\}$$

Ex $\longrightarrow$ $y = e^{mx}$ (نفرض) (كتخمين (لا تتغير))

$$y' = me^{mx} \Longrightarrow y'' = m^2 e^{mx}$$

$$am^2 e^{mx} + bme^{mx} + ce^{mx} = 0$$

$$e^{mx}[am^2 + bm + c] = 0$$

$$y_1 = e^{m_1 x}, \quad y_2 = e^{m_2 x}$$

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Provided that $w(y_1, y_2) \neq 0$

$$w(y_1, y_2) = y_1 y_2' - y_2 y_1' = e^{m_1 x} m_2 e^{m_2 x} - e^{m_2 x} m_1 e^{m_1 x}$$

$$= (m_2 - m_1) e^{(m_1 + m_2)x} \neq 0$$

except when $m_1 = m_2$

مثال :- ايجاد حل تام للمعادلة التالية :-

$$y'' + 7y' + 12y^0 = 0$$

تأخذ m بدل المشتقة

$m^0 = 1$

$$m^2 + 7m + 12 = 0$$

$$(m+3)(m+4) = 0$$

$\longrightarrow$ y, m

$$m_1 = -3, \quad m_2 = -4$$

$$y(x) = c_1 e^{-3x} + c_2 e^{-4x}$$

مثال ايجاد حل تام للمعادلة التالية :-

$$y'' + 2y' + 5y = 0$$

$$m^2 + 2m + 5 = 0$$

$\rightarrow$ -B+-(b^2-4AC)^0.5/2*A
Ax^2+Bx+c=0

$$m_{1,2} = \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$m_{1,2} = -1 \pm 2\sqrt{-1} = -1 \pm 2i$$ Conjugated Complex roots

$$y(x) = c_1 e^{(-1+2i)x} + c_2 e^{(-1-2i)x}$$

$m_{1,2} = p \pm qi$

$y(x) = c_1 e^{(p+qi)x} + c_2 e^{(p-qi)x}$

$= e^{px}[c e^{qix} + c_2 e^{-qxi}]$ $\quad \left\{ \begin{array}{l} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{array} \right.$

$= e^{px}[c_1(\cos qx + i\sin qx) + c_2(\cos qx - i\sin qx)]$

$(c_1 - c_2)i$

$= e^{px}[(c_1 + c_2)\cos qx + \quad \sin q x]$

$$\boxed{y(x) = e^{px}[A\cos qx + B\sin qx]}$$

sol $\quad p = -1 \quad q = 2 \Longrightarrow y(x) = e^{-x}[A\cos 2x + B\sin 2x]$

مثال : إيجاد حل تام للمعادلة التالية :

$y'' - 2ry' + r^2 y = 0 \qquad r = \text{constant}$

$m^2 - 2rm + r^2 = 0$

$(m-r)^2 = 0$

$m_1 = m_2 = r \Longrightarrow y_1 = e^{m_1 x} = e^{rx}$

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx$$

$p(x) = -2x \longrightarrow \int p(x)dx = \int -2r\,dx = -2rx$

$e^{-\int p(x)dx} = e^{2rx}$

$y_2 = e^{rx} \int \frac{e^{2rx}}{e^{2rx}} dx = x e^{rx}$

قاعدة : في حالة $m_1 = m_2$ فالحل الثاني $y_2 =$ الحل الاول y_1 مضروب في x

$y_2 = x y_1$

$y(x) = c_1 e^{m_1 x} + c_2 x e^{m_1 x} \qquad \text{when } m_1 = m_2$

$= c_1 e^{rx} + c_2 x e^{rx}$

① y Global ② Local

Application of second order D.E

⊛ Special types of second order D.E

Certain type of second order D.E such as :-

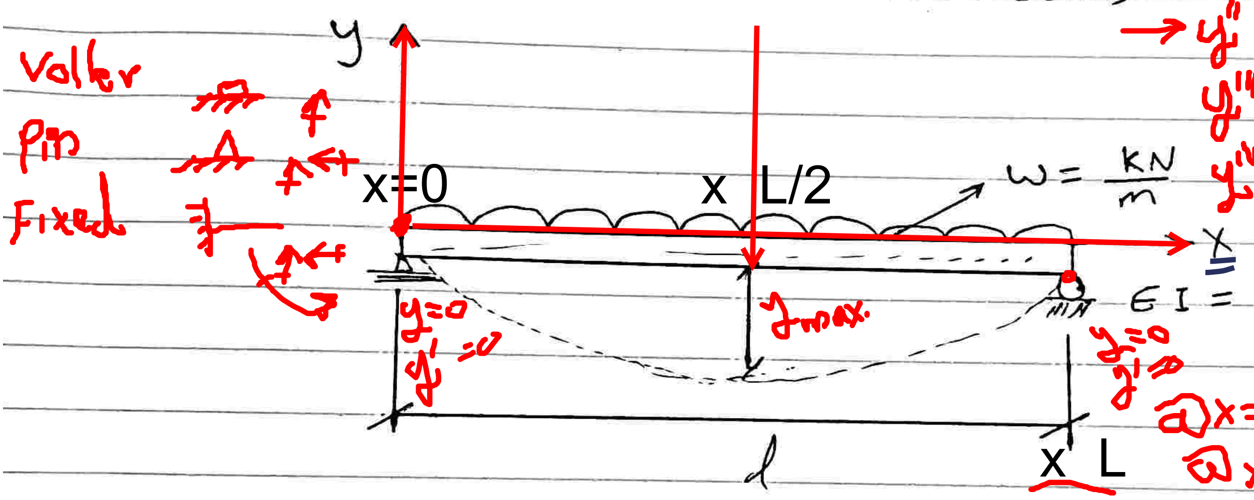
$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = f(x) \quad \text{---- eq (1)}$$

Can be reduced to 1st order equations by a suitable change of variables:-

→ SSB ⇒ reac. = $\frac{\text{Load}}{2}$

y = defl.
y' = slope
y'' = M
y''' = V
y'''' = Load

a) Deflection of beam (determinate beam)



@ x=0, y=0 —— ①
@ x=L, y=0 —— ②

M=Fd

$$\boxed{\text{The equation } y'' = \frac{M}{EI}}$$

كل مائل يحلل
كل منتشر يركز

$\Sigma F_x = 0$
$\Sigma F_y = 0$ → reaction
$\Sigma M = 0$

ملاحظات ①

1. لغرض أخذ مقطع في الـ Beam لتحديد العزم (moment) لابد من تحديد نقطة الأصل للمحورين حيث ان محور X داخلاً موازي للعتبة (beam) ومحور y عمودي على العتبة، والى الاعلى موجب والـ X باتجاه اليمين موجب.

2. بعد تحديد المحاور وإيجاد ردود الافعال يتم اخذ مقطع على بعد (X) من نقطة الأصل [نقطة التقاء المحورين X, y] وإيجاد العزم على بعد X والانتباه الى الإشارة ثم تطبيق المعادلة:-

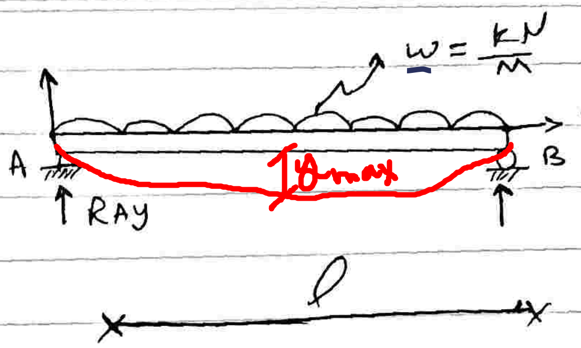
$$y'' = \frac{M(x)}{EI} \rightarrow \text{حفظ}$$

E : modulus of elasticity
I : moment of Inertia.

Ex

$\curvearrowright + \sum M_B = 0$

$R_{Ay} * l - w * l \times \frac{l}{2} = 0$

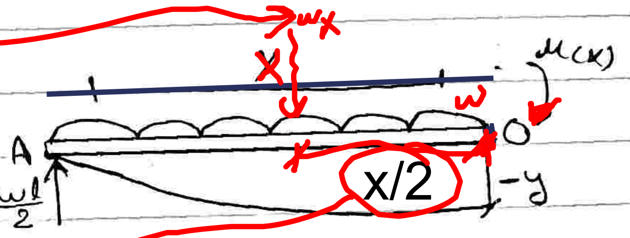


(1) $R_{Ay} = \frac{wl}{2}$

$M = F.d$

(2) $+\sum M_O = 0$ عكس عقرب الساعة

$\frac{wl}{2} \cdot x + M - wx \cdot \frac{x}{2} = 0$



(max. deflection)

$x=0$, $y=0$, $y''=0$

$x = \frac{l}{2}$, $y' = 0$

$$\boxed{M = \frac{wx^2}{2} - \frac{wlx}{2}}$$

(3) $\therefore y'' = \frac{-M_x}{EI}$

$\int y''$, $\int y'$, y = Deflection

C_1, C_2

$$y'' = -\left(\frac{\frac{wx^2}{2} - \frac{wl}{2}}{EI}\right)$$

$$\int y'' = \int \frac{\frac{wlx}{2} - \frac{wx^2}{2}}{EI}$$

$$y' = \int \frac{wlx}{2EI}\,dx - \int \frac{wx^2}{2EI}\,dx$$

$\int 3\,dx = 3x$

$3x' = 3$

$$y' = \frac{wl}{4EI}x^2 - \frac{w}{6EI}x^3 + C_1$$

$y\ \ EI \downarrow \int$

$$\boxed{y = \frac{wl}{12EI}x^3 - \frac{w}{24EI}x^4} + C_1 x + C_2$$

$C_1 = \frac{-wl^3}{24EI}$

$C_2 = 0$

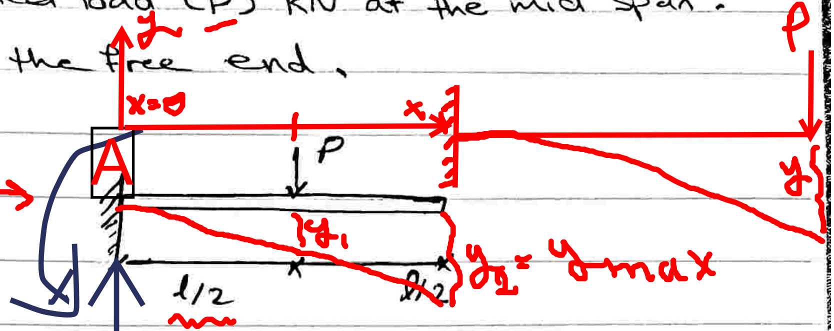
B.C to evaluate C_1 and C_2

B.C (1) $\to x = 0$, $y = 0$ deflection

B.C (2) $\to x = l/2$, $y' = 0$ Slope

$y = \frac{wl}{12EI} \times \left(\frac{L}{2}\right)^3 - \frac{w}{24EI}\left(\frac{L}{2}\right)^4$

e.g ① A Cantliver beam of length (l) and uniform section (EI) is being located with constructed load (P) kN at the mid span. Determine the deflection at the free end.



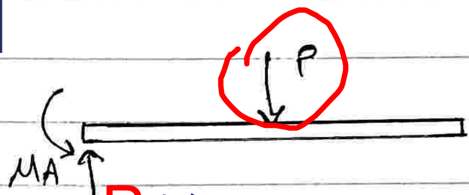
Solution:- $\sum F_x = 0 \Rightarrow A_x = 0$

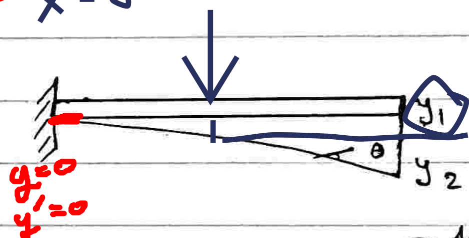
$\sum F_y = 0 \Rightarrow A_y = P$

$\curvearrowright + \sum M_A = 0$

$P \cdot \frac{l}{2} - M_A = 0$

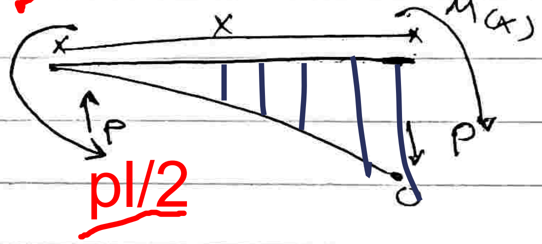
① $\boxed{M_A = \frac{Pl}{2}}$ ve a X





فأخذ قطع عامة X (يكون قبل P)

② $\curvearrowright + \sum M_o = 0$ $\left(\frac{Pl}{2}\right)$, P+



$M_x - \frac{Pl}{2} + Px = 0 \Rightarrow M_x =$

③ $\therefore y'' = \frac{-M_x}{EI} \Rightarrow y'' = \frac{-(Pl/2 - Px)}{EI}$

$\int y'' = \int \left(\frac{Px}{EI} - \frac{Pl}{2EI}\right) dx$

$\int 3x = \frac{3x^2}{2}$, $\int 3\,dx$

$y' = \frac{Px^2}{2EI} - \frac{Pl}{2EI} x + \cancel{C_1}$

$\int \downarrow$ $y = \frac{Px^3}{6EI} - \frac{Plx^2}{4EI} + \cancel{C_1 X} + \cancel{C_2}$

④ B.C① $\left.\begin{matrix} x = 0 \\ y = 0 \end{matrix}\right\}$ $0 = 0 - 0 + 0 + C_2 \Rightarrow \boxed{C_2 = 0}$

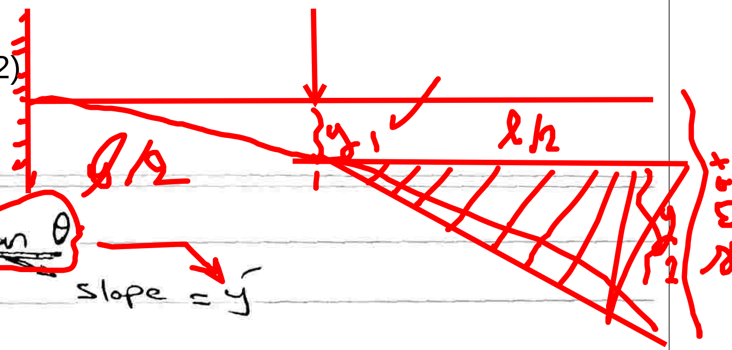
B.C② $\left.\begin{matrix} x = 0 \\ y' = 0 \end{matrix}\right\}$ $0 = 0 - 0 + C_1 \Rightarrow \boxed{C_1 = 0}$

$\left(\frac{l}{2}\right)^3 = \frac{l^3}{2^3} \rightarrow 8$ $\left(\frac{l}{2}\right)^2 = \frac{l^2}{2^2} \rightarrow 4$

$\therefore y = \frac{Px^3}{6EI} - \frac{Pl}{4EI}x^2$

$y_1(l/2) = \frac{Pl^3}{48} - \frac{Pl^3}{16EI} = \frac{-2Pl^3}{48EI} = \frac{-Pl^3}{24EI}$

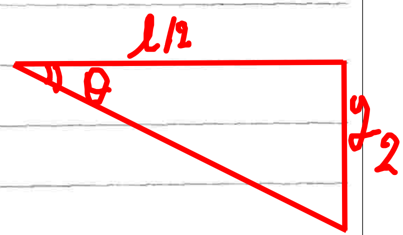
$\frac{1}{48} - \frac{1}{16}_{(3)} = \frac{1-3}{48} = -\frac{1}{24}$



* Max. Deflection

$$\Rightarrow \tan\theta = \frac{y_2}{l/2} \longrightarrow y_2 = \frac{l}{2}\tan\theta \rightarrow \text{slope} = y'$$

$$\therefore y_2 = \frac{l}{2}\, y'\left(\frac{l}{2}\right)$$



$$y'\left(\frac{l}{2}\right) = \frac{Pl^2}{8EI} - \frac{Pl^2}{4EI} \Rightarrow \frac{1}{8_{(1)}} - \frac{1}{4_{(2)}} = \frac{1-2}{8}$$

$$= \frac{-Pl^2}{8EI} = \tan\theta$$

$$\therefore y_2 = \frac{l}{2} \cdot \frac{-Pl^2}{8EI} = \frac{-Pl^3}{16EI}$$

deflection @ free end.

$$y_{end} = y_1 + y_2$$

$$= -\left[\frac{Pl^3}{24EI} + \frac{Pl^3}{16EI}\right] = -\left[\frac{5}{48}\right]\frac{Pl^3}{EI}$$

Ex(2) Find the deflection equation :-

Solution

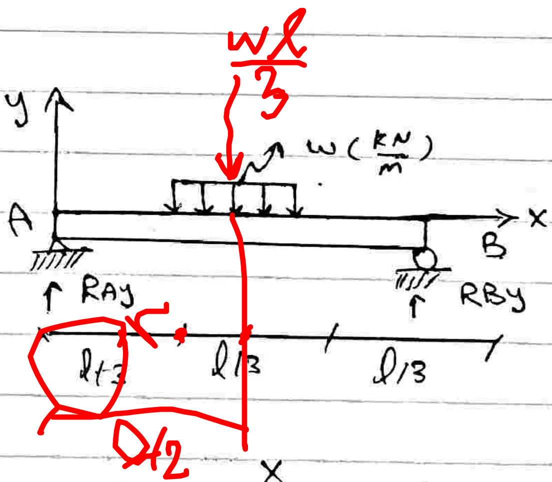
1

$$\overset{\curvearrowright}{+} \sum M_B = 0$$

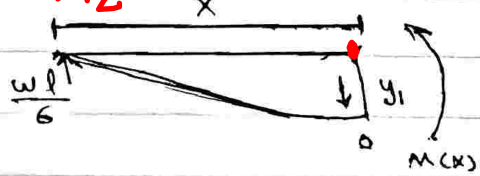
$$R_A(l) - w\left(\frac{l}{3}\right) \times \frac{l}{2} = 0$$

$$R_A = \frac{wl}{6}$$

يكمل الحل بعدين



a) $0 \leq x \leq 3$



$$\overset{\curvearrowright}{+} \sum M_o = 0$$

2

$$\frac{wl}{6}(x) - M(x) = 0$$

$$M(x) = \frac{wlx}{6}$$

$$\because y'' = \frac{M}{EI}$$

$$y'' = \frac{-wlx/6}{EI} \Longrightarrow \int y'' = \int \frac{wlx}{6EI}\,dx$$

$$y' = \frac{wl}{12EI}x^2 + C_1$$

$$\int\int dx \quad y_1 = \frac{wl}{36EI}x^3 + C_1 x + C_2$$

b) $\frac{l}{3} \leq x \leq \frac{l}{2}$

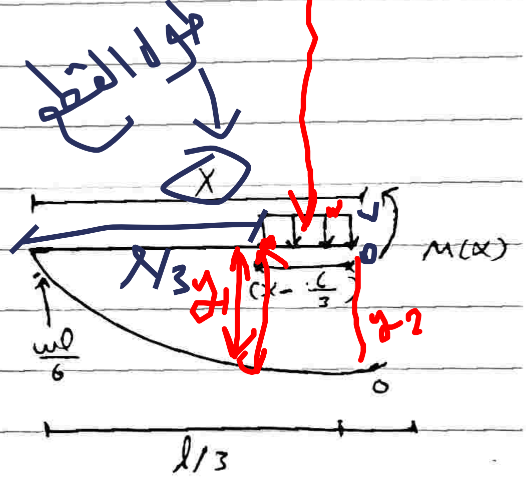
$\curvearrowleft + \sum M_o = 0$

$$\frac{wl}{6}(x) - M(x) - \frac{w\left(x-\frac{l}{3}\right)^2}{2} = 0$$

$$M(x) = \frac{wlx}{6} - \frac{w\left(x^2 - \frac{2}{3}x + \frac{l^2}{9}\right)}{2}$$

$$\Rightarrow M(x) = \frac{wlx}{6} - \frac{wx^2}{2} + \frac{wlx}{3} - \frac{wl^2}{18}$$

$$\Rightarrow M(x) = \frac{wlx}{2} - \frac{wx^2}{2} - \frac{wl^2}{18}$$



$$\because y'' = \frac{M(x)}{EI}$$

$$\int y'' = \int \left(\frac{wlx}{2EI} - \frac{wx^2}{2EI} - \frac{wl^2}{18EI}\right)dx$$

$$\int y' = \int \left(\frac{wlx^2}{4EI} - \frac{wx^3}{6EI} - \frac{wl^2x}{18EI} + C_3\right)dx$$

$$y_2 = \frac{wlx^3}{12EI} - \frac{wlx^4}{24EI} - \frac{wl^2x^2}{36EI} + C_3x + C_4$$

B.C (1) $\to x = 0,\ y_1 = 0 \Longrightarrow C_2 = 0$

B.C (2) $\to x = 0,\ y_2 = 0 \longrightarrow C_4 = 0$

B.C (3) $\to x = l/3,\ y_1 = y_2 \Longrightarrow$

B.C (4) $\to x = l/3,\ y'_1 = y'_2 \Longrightarrow$

} C_1 and C_3

The non homogenous Differential Equation

$ay'' + by' + cy = f(x)$

$y(x) = \underbrace{c_1 y_1 + c_2 y_2}_{y_c} + Y$ — Particular integral الحل الخاص

y_c — Complementary function الدالة المتممة

Y → method of undetermined Coefficients (طريقة خاصة بنوع معين من الدوال)

Y → method of variation of parameter.

Method of Undetermined Coefficients

e.g. $y'' + 4y' + 3y = 5e^{2x}$ → Ae^{2x} — y_p

① $m^2 + 4m + 3 = 0$

$(m+1)(m+3) = 0 \Rightarrow m_1 = -1,\ m_2 = -3$

$\boxed{y_c(x) = c_1 e^{-x} + c_2 e^{-3x}}$ — y_c مبتدأ :ok

$y(x) = y(c) + Y$

② $Y = Ae^{2x}$ → $y = y_p$ (الخاص)

$Y' = 2Ae^{2x}$

$Y'' = 4Ae^{2x}$

بقد رمشتقات السؤال

$e^{2x} = 2e^{2x}$

③ $4Ae^{2x} + 4(2Ae^{2x}) + 3Ae^{2x} \stackrel{?}{=} 5e^{2x}$

$15Ae^{\cancel{2x}} = 5e^{\cancel{2x}}$

$15A = 5$

$A = \frac{1}{3}$ ✓ → $y_p = Ae^{2x}$

$$Y_p = \frac{e^{2x}}{3}$$

$$y_h = \underbrace{c_1 e^{-x} + c_2 e^{-3x}}_{y_c} + \underbrace{\frac{e^{2x}}{3}}_{y_p}$$

$y_c = c_1 e^{-x} + c_2 e^{-3x}$

$\boxed{e.g \quad \overset{2}{y''} + \overset{y_c}{4y'} + 3y = \overset{y_p}{5 \sin 2x}}$

$\sin(2x)$

$\cos 2x \cdot 2$

$2\cos 2x$

assume y_p = $Y = A \sin 2x \Longrightarrow Y' = 2A\cos 2x \longrightarrow Y'' = -4 \sin 2x$

option 1

$-4A\sin 2x + 4(2A\cos 2x) + 3A\sin 2x \overset{??}{=} 5 \sin 2x$

$-A\sin 2x + 8A\cos 2x \overset{?}{=} 5 \sin 2x + 0 \cos 2x$

مساواة $\sin 2x$

$-A = 5 \Longrightarrow A = -5$

$8A = 0 \Longrightarrow A = 0$??

option 2

$y_p = Y = A\sin 2x + B\cos 2x$ (assumed)

$Y' = 2A\cos 2x - 2B\sin 2x$

$Y'' = -4\sin 2x - 4B\cos 2x$

تعويض بالمعادلة $\left[-4A\sin 2x - 4B\cos 2x\right]_{y''} + \left[4[2A\cos 2x - 2B\sin x]\right]_{4y'} + \left[3[A\sin 2x + B\cos 2x]\right]_{3y_p} \overset{?}{=} 5\sin 2x$

$[-4A - 8B + 2A]\sin 2x + [-4B + 6A + 3B]\cos 2x \overset{?}{=} 5\sin 2x$

$(-A - 8B)\sin 2x + [8A - B]\cos 2x = 5\sin 2x + 0\cos 2x$

مساواة $\sin 2x$

$$\left.\begin{aligned} -A - 8B &= 5 \\ 8A - B &= 0 \end{aligned}\right\} \Longrightarrow A = -\frac{1}{13}, \quad B = \frac{-8}{13}$$

$$Y_p = -\frac{\sin 2x + 8\cos 2x}{13}$$

$$y_h(x) = \underbrace{C_1 e^{-x} + C_2 e^{-3x}}_{y_c} + \underbrace{\frac{\sin 2x + 8\cos 2x}{13}}_{y_p}$$

عندما تكون الدوال في الجهة اليمنى تمتلك عدد محدد من المشتقات المستقلة خطياً فبالإمكان تكوين الحل الخاص Y بأخذ جميع المشتقات المستقلة خطياً ومن ضمنها الدالة و ضرب كل واحد منها بثابت اختياري وجمعهم خطياً .

وهذه الدوال هي :

e^{kx} , $\sin kx$, $\cos kx$, x^n ($n =$ عدد صحيح موجب) , k

وكل شيء ناتج عن جمع او ضرب او طرح بين الدوال المذكورة

y_c $\quad$ y_p $\quad$ e^{-3x} $\quad$ $=3e^{-3x}$

$m = y'$

$m^0 = y = 1$

$-3x = -3$

e.g. $\quad y'' + 4y' + 3y = 5e^{-3x}$

y_L

$$m^2 + 4m' + 3 = 0$$

$$(m+1)(m+3) \Rightarrow m_1 = -1 \quad m_2 = -3$$

$$y = c1e^{-x} + c_2 e^{-3x}$$

$$c1e^{m1x}+c2e^{m2x}$$

$$Y(x) = C_1 e^{-x} + C_1 e^{-3x} + Y_p$$

$$Y = A e^{-3x} \Rightarrow Y' = -3Ae^{-3x} \Rightarrow Y'' = 9Ae^{-3x}$$

$$9Ae^{-3x} + 4(-3Ae^{-3x}) + 3Ae^{-3x} \overset{??}{=} 5e^{-3x}$$

$$\Rightarrow y'' + 4y' + 3y = 5e^{kx}$$

$$Y = Ae^{kx} \longrightarrow Y' = kAe^{kx} \longrightarrow Y'' = k^2 Ae^{kx}$$

$$k^2Ae^{kx} + 4kAe^{kx} + 3Ae^{kx} = 5e^{kx}$$

$$A(k^2+4k+3) = 5$$

$$A = \frac{5}{k^2+4k+3}$$

$$Y = \frac{5e^{kx}}{k^2+4k+3}$$

$$y = C_1 e^{-x} + C_2 e^{-3x} + Y$$

$$= C_1 e^{-x} + \left(\bar{C}_2 - \frac{5}{k^2+4k+3}\right) e^{-3x} + Y$$

$$\acute{Y} = Y - \frac{5e^{-3x}}{k^2+4k+3} = \frac{5e^{kx} - 5e^{-3x}}{k^2+4k+3}$$

$$\acute{Y} \qquad \frac{5xe^{kx}}{2k+4} = \frac{5xe^{-3x}}{-2}$$

* نتيجة : إذا حدث تكرار بين الدالة (y_c) المتممة والحل الخاص المفترض (Y). الدالة
هذا التكرار بالضرب في x^n بحيث ان (n) هو اصغر عدد صحيح موجب بعد التكرار
(نخلص من التكرار في x^n ، n اصغر عدد صحيح موجب يبعد التكرار)
وللمثال المعطى ⟵

$$Y = Ae^{-3x} \Longrightarrow Y = Axe^{-3x}$$

$$y' = A(-3xe^{-3x}(16) + e^{-3x})$$

$y_c + y_p$

Find the particular solution y_p

Find the complete solution of the below ODE)

Find the complementary solution y_c

Example :— Y_1 Y_2 Y_3

جد حلاً تاماً للمعادلة التفاضلية الآتية :-

$y'' + 9y = 3x^2 + 4x + 7$

$m^2 + 9 = 0 \Rightarrow m^2 = -9 \Rightarrow m_{1,2} = \mp 3i$

$\sqrt{9 \times -1} \Rightarrow \sqrt{9}\sqrt{-1}$

جذرين مركبين وتخيلين

$y_c(x) = A\cos 3x + B\sin 3x$

$= \frac{3}{x} + 5$

$y_p = Ax^3 + Bx^2 + Cx + D$

$Y_1 = a_0 x^2 + a_1 x + a_2$

$Y_2 = b_1 x + b_2$

$Y_3 = c_2$

$Y = Y_1 + Y_2 + Y_3 = a_0 x^2 + (a_1 + b_1)x + (a_2 + b_2 + c_2)$

إذن $Y_p = A_0 x^2 + A_1 x + A_2$

$Y' = 2A_0 x + A_1$

$Y'' = 2A_0$

$2A_0 + 9(A_0 x^2 + A_1 x + A_2) = 2x^2 + 4x + 7$

تساوي $9A_0 x^2 + 9A_1 x + 2A_0 + 9A_2 = 2x^2 + 4x + 7$

x^2 $9A_0 = 2 \Rightarrow A_0 = \frac{2}{9}$

x $9A_1 = 4 \Rightarrow A_1 = \frac{4}{9}$

$2A_0 + 9A_2 = 7 \Rightarrow A_2 = \frac{59}{81}$

$Y = \frac{2}{9}x^2 + \frac{4}{9}x + \frac{59}{81}$

y_p

$y(x) = y_c + Y_p = A\cos 3x + B\sin 3x + \left[\frac{18x^2 + 30x + 59}{81}\right]$

Find the complete solution of the below ODE)

مثال :- جد حل المعادلة الآتية بحيث أن

y_c y_p

$y'' - 4y' + 4y = 0$

$y(t=0) = 3$ (1) , $y'(t=0) = 4$ (2)

$m^2 - 4m + 4 = 0$

$(m+4)(m+1)$

$(m-2)^2 = 0$

$m_1 = m_2 = 2$ ✓

B.C

$\boxed{y(t) = C_1 e^{2t} + C_2 t e^{2t}}$ → $y(t)$

$-4m$

$1m$

$3 = C_1(1) + C_2(0)$

$y'(t) = 2C_1 e^{2t} + C_2 e^{2t} + 2C_2 t e^{2t}$

$y(0) \boxed{= 3 = C_1}$ ✓

$3 = C_1$

$y'(t) = 2C_1 e^{2t} + C_2(2te^{2t} + e^{2t})$

$4 = 2(3) + C_2(0+1) \Longrightarrow \boxed{C_2 = -2}$

$\therefore \boxed{y(t) = 3e^{2t} - 2te^{2t}}$

Find the complete solution of the below ODE)

مثال :- حل المعادلة التالية بحيث ان ؟ ⟵

$(4D^2 + 16D + 17)y = 0$

$\dfrac{-B \mp \sqrt{B^2 - 4AC}}{2A}$

$Ax^2 + Bx + C = 0$

(1) $y(t=0) = 1$

(2) $y(t=\pi) = 0$

sol → $4D^2 + 16D + 17 = 0$

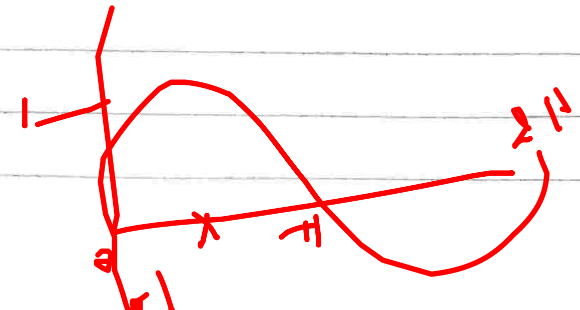
$D_{1,2} = -2 \pm \dfrac{i}{2}$ → e (AcosBx+BsinBx)

$y(t) = e^{-2t}[A\cos 0.5t + B\sin 0.5t]$

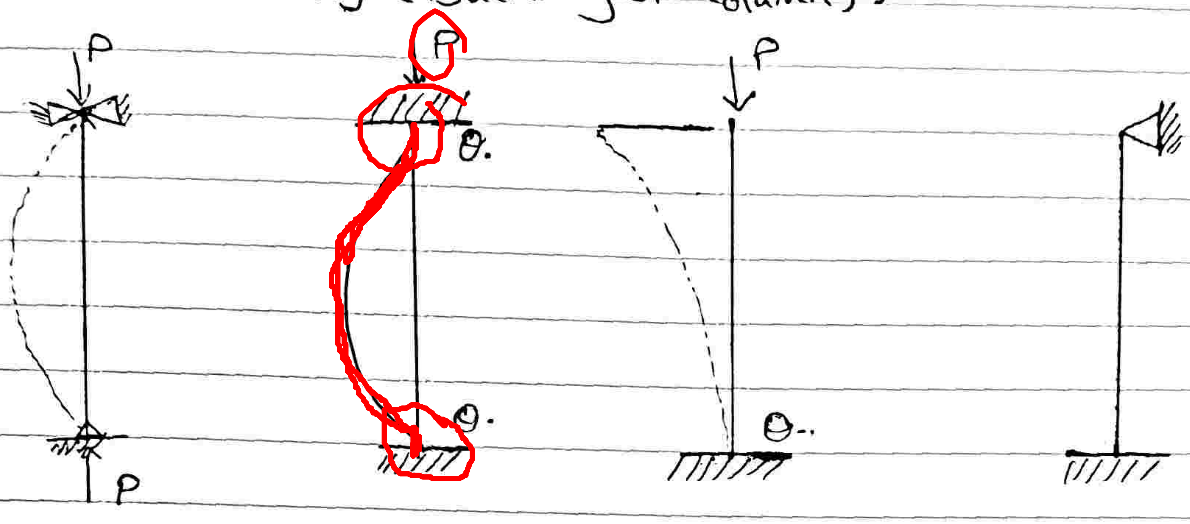
$y(0) = \boxed{1 = A}$

$y(\pi) = 0 = e^{-2\pi}[0 + B] \Longrightarrow Be^{-2\pi} \Longrightarrow \boxed{B = 0}$

$\boxed{y(t) = e^{-2t}\cos\dfrac{t}{2}}$



b) Elastic stability (Buckling of Column) :-



hinge-hinge Ends | Fixed-Fixed Ends | Free-Fixed Ends | hinge-Fixed Ends

y = deflection , $\frac{dy}{dx}$ = slope (θ) , $\frac{d^2y}{dx^2}$ = moment

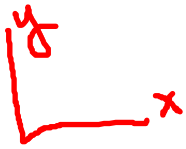
$\frac{d^3y}{dx^3}$ = shear , $\frac{d^4x}{dy^4}$ = load

Note :- x-axis parallel to the column
y-axis orthogonal to the column

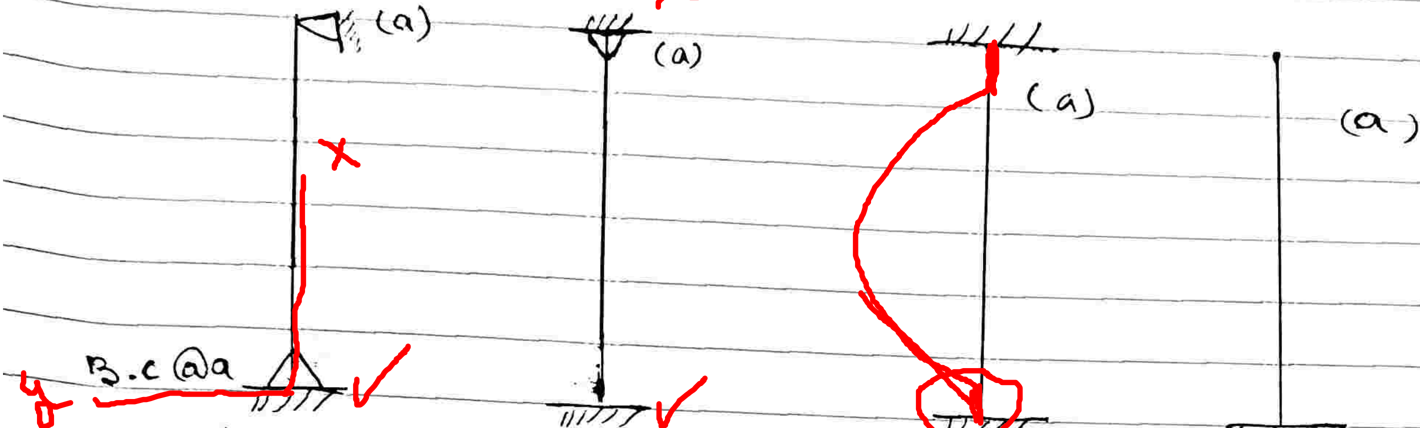
ملاحظة // عندما نحنا ، المحور (y) سيكون دائماً عمودي على العمود باتجاه التشوه
ونقطة الأصل تكون في الموقع الذي يكون أقل ردود أفعال

Govering diff. equation $y'' = \frac{M(x)}{EI}$

Where :- E : modulus of elasticity
I : moment of Inertia



⊛ Boundary Condition (B-C)



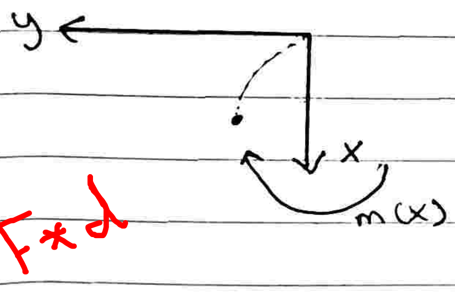
B.c @a	B.C @a	B.C @a	B.C @a
$\Delta = y = 0$	$y = 0$	$y = 0$ ✓	$M = y'' = 0$
$M = y'' = 0$	$y' = 0$	$y' = 0$ ✓	$V = y''' = 0$

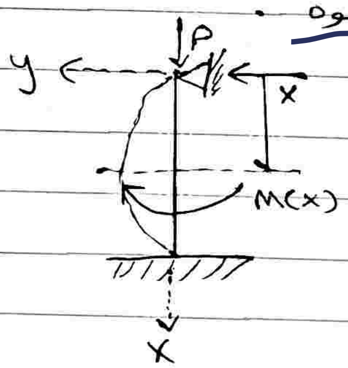
خطوات الحل :

١- تحدد نقطة الأصل

٢- تحدد المحاور y , x او تحدد ردود الأفعال

٣- نأخذ قطع من جهة نقطة الأصل ونقوم بحساب العزم ويكون دائماً العزم باتجاه التشوه .





$M = F * d$

P_{cr}

* Ex. ① Find the critical load can be applied on hinged - hinged Column before failure due to buckling ?

① $\curvearrowleft + \sum M_B = 0$

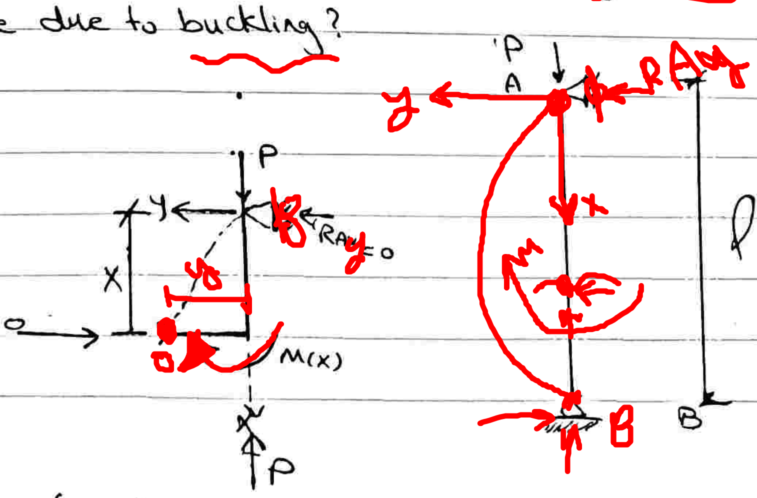
$R_{Ay}(L) = 0$

$R_{Ay} = 0$

② $\curvearrowleft + \sum M_o = 0$

$M(x) + P(y) = 0$

$\boxed{M(x) = -P \cdot y}$



(3) $\therefore y'' = \frac{\cancel{-}Mx}{EI} \Longrightarrow y'' = \frac{-P \cdot y}{EI} \Longrightarrow \boxed{\underbrace{y''}_{y_c} + \frac{P}{EI} \underbrace{y = 0}_{y_p}}$

O.D.E (2nd order) (1st degree) (homogenous) (linear)

* let $\frac{P}{EI} = \lambda^2$

$y'' + \lambda^2 y = 0 \Longrightarrow m^2 + \lambda^2 = 0$ → $\alpha \pm Bi$

$m^2 = -\lambda^2$

$m_{1,2} = \pm\lambda i$ → $\alpha = 0$, $B = \lambda$

$y(x) = \cancel{e^{0}}^{\,1} (C_1 \cos \lambda x + C_2 \sin \lambda x)$

$= C_1 \cos \lambda x + C_2 \sin \lambda x$

B.C (1): $x = 0$, $y = 0 \Longrightarrow 0 = C_1(1) + C_2(0) \Longrightarrow \boxed{C_1 = 0}$

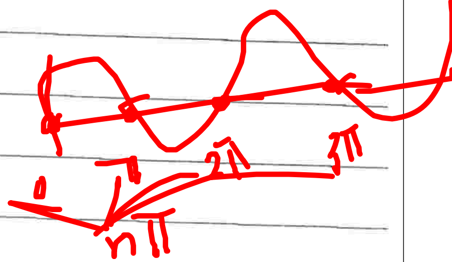
B.C (2): $x = l$, $y = 0 \Longrightarrow 0 = C_2 \sin \lambda L$

$\therefore \sin \lambda L = 0$ $\qquad 0 \neq P$

$\lambda L = \sin^{-1} 0$ $\qquad$ الصفر يهمل من ضمن الزوايا لأنه يعطي قوة مقدارها صفر وذلك لان $\sin^{-1}$ يتغير

$\lambda L = n\pi$ where $n = 1, 2, 3, 4$

$\lambda = \frac{n\pi}{L}$ $\qquad y(x) = \sum_{n=1}^{\infty} C_{2n} \sin \frac{n\pi x}{l}$



but $\frac{P}{EI} = \lambda^2$

$\lambda = \sqrt{\frac{P}{EI}} \longrightarrow \frac{n\pi}{l} = \sqrt{\frac{P}{EI}}$

$\frac{P_{cr}}{EI} = \frac{n^2\pi^2}{l^2} \Longrightarrow \boxed{P_{cr} = \frac{n^2\pi^2}{l^2} \cdot EI}$ (Euler's Buckling load)

(buckling equation)

ملاحظة : لو كان المطلوب هو ال λ فان المطلوب ايجاده هو

$\lambda = \frac{n\pi}{l}$

but $\lambda = \sqrt{\frac{F}{EI}}$

$\therefore y(x) = \sum_{n=1}^{\infty} C_{2n} \sin \frac{n\pi}{l}$

Higher order Linear D.E with Constant Coefficient :-

A differential equation can be consider as (H.O.D.E) if and only if the order is greater than or equal three, $(n \geq 3)$ the standard form of such equation are :-

$$a\frac{d^n y}{dx^n} + b\frac{d^{n-1} y}{dx^{n-1}} + C\frac{d^{n-2} y}{dx^{n-2}} + \text{----} = 0 \text{ or } \neq 0$$

Constant or $f(x)$

$= 0$ homogenous $y(x) = y_h$

$\neq 0$ non homogenous $y(x) = y_c + y_P$

y_c: u.c $\quad$ y_P: v.P

ⓐ → Homogenous linear D.E with Constant Coeff. :-

$y(x) = y_h$

After we get the root $m_1, m_2, m_3, \text{---} m_n$

Case written solution :-

Case ① if $m_1 \neq m_2 \neq m_3 \text{ ---} \neq m_n$ (Real)

$$y_n = C_1 e^{m_1 \cdot x} + C_2 \cdot e^{m_2 \cdot x} + C_3 \cdot e^{m_3 x} + \text{-----} + C_n e^{m_n x}$$

Case ② If $m_1 = m_2 = m_3 = \text{----} = m_n$ (real)

$$y_n = e^{m_1 x}\left(C_1 X^P + C_2 X^{P-1} + C_3 X^{P-2} + \text{---}\right)$$

P = no. of root = iteration -1 $\quad$ عدد التكرارات

ex $\quad \dfrac{d^6 y}{dx^6} - \dfrac{d^5 y}{dx^5} = 0$

$m^6 - m^5 = 0 \Longrightarrow m^5 (m-1) = 0 \qquad P = 5-1 = 4$

$m^5 = 0 \qquad\qquad m - 1 = 0$

$m_1 = m_2 = m_3 = m_4 = m_5 = 0 \qquad\qquad m_6 = 1$

If F(x) has a term of	The Expression for (yp)
any constant $(\ldots, -2, \frac{-1}{2}, \frac{1}{3}, 5, \ldots)$	A, Ex / $10 \Rightarrow yp = A$ Ex / $\frac{-1}{2} \Rightarrow yp = A$
$e^{m\cdot x}$	$A\cdot e^{m\cdot x}$ Ex / $2e^{x} \Rightarrow yp = Ae^{x}$ Ex / $\frac{-4}{e^{2x}} \Rightarrow 4e^{-2x} \Rightarrow yp = Be^{-2x}$
$\sin kx$ or $\cos kx$ or both	$A\cos kx + B\sin kx$ Ex/ $\sin 3x$ $yp = A\cos 3x + B\sin 3x$ Ex/ $2\cos x + 10\sin 2x$ $(yp = A\cos x + B\sin x)$ $(yp = A\cos 2x + B\sin 2x)$ $yp = A(\cos x + \cos 2x) + B(\sin x + \sin 2x)$
polynomial $(x+1, x^2-x, x^2+x+1)$	$Ax^{n} + Bx^{n-1} + \text{-----}$
$x^{n}\cdot e^{mx}\cdot\sin kx$ or $x^{n}e^{mx}\cos kx$ $x^{n}\cdot e^{mx}$ or $x^{n}\sin kx$ or $x^{n}\cos kx$	$(Ax^{n} + Bx^{n-1} \ldots)e^{mx}\sin kx + (Ex^{n} + Fx^{n-1})e^{mx}\cos kx$ $(Ax^{n} + Bx^{n-1})e^{mx}$ $(Ax^{n} + Bx^{n-1} \ldots)\sin kx + (Sx^{n} + Dx^{n-1} \ldots)\cos kx$.

$$y_h = C_1 e^{x} + e^{0x}[C_2 x^4 + C_3 x^3 + C_4 x^2 + C_5 x + C_6]$$
$$= C_1 e^{x} + C_2 x^4 + C_3 x^3 + C_4 x^2 + C_5 x + C_6$$

Case (3) If $m_{1,2} = \alpha_1 \mp \beta_1 i = m_3, m_4 = m_5, m_6 ----$

$$y_h = e^{\alpha x}[C_1 \cos \beta x + C_2 \sin \beta x] + x e^{\alpha x}[C_3 \cos \beta x + C_4 \sin \beta x] + x^2 e^{\alpha x}[C_5 \cos \beta x + C_6 \sin \beta x] + ----$$

Case (4) If $m_1 = \alpha_1 \mp \beta_1 i, m_2 = \alpha_2 \mp \beta_2 i, --- m_n = \alpha_n \mp \beta_n i$

$$y_h = e^{\alpha_1 x}[C_1 \cos \beta_1 x + C_2 \cos \beta_1 x] + e^{\alpha_2 x}[C_3 \cos \beta_2 x + C_4 \cos \beta_2 x]$$

Case (5) If $m_1 = 5, m_2 = m_3 = m_4 = m_5 = -2,$
$m_6 = m_7 = \mp i, m_{8,9} = \mp i, m_{10}, m_{11} = 2 \mp 3i$

$$y_h = C_1 e^{5x} + e^{-2x}(C_2 x^3 + C_3 x^2 + C_4 x + C_5) + e^{0x}(C_6 \cos x + C_7 \sin x) + x e^{0\cdot x}(C_8 \cos x + C_9 \sin x) + e^{2x}(C_{10} \cos 3x + C_{11} \sin 3x).$$

ملاحظة لغرض تحديد جذور المعادلة (قيم m التي تجعل الطرف الأيمن يساوي صفر) تستخدم الطرق التالية:

(1) طريقة العامل المشترك، عدم وجود y في المعادلة

ex. solve $y'''' - y'' = 0$

$m^4 - m^2 = 0 \Rightarrow m^2(m^2 - 1) = 0$

$m^2(m-1)(m+1) = 0$

either $m_1 = m_2 = 0, m_3 = 1, m_4 = -1$

X_1, X_2

$$y_h = e^{0\cdot x}[C_1 x + C_2] + C_3 e^{x} + C_4 e^{-x}$$
$$y(x) = y_c = C_1 x + C_2 + C_3 e^{x} + C_4 e^{-x}$$

$Ax^2 + Bx + c = 0$

(2) في حالة وجود تناظر باسم المتغير بحيث ينطبق عليه الدستور

ex. $m^4 + 2m^2 - 3 = 0$
$3m^6 + 3m^3 + 10 = 0$
$10m^8 + m^4 - 2 = 0$

ينطبق عليه الدستور

-B+-(b^2-4AC)/2A
Ax^2+Bx+C=0

$(x^2+1)(x^2+9)=0$

Ex. Solve $y'''' + 10y'' + 9y = 0$

Sol. $m^4 + 10m^2 + 9 = 0$, $a=1$, $b=10$, $c=9$

$m^2 = \frac{-10 \mp \sqrt{100-36}}{2} = -5 \mp \frac{8}{2} = -5 \mp 4$

$(m^2+9)^2(m^2+1)=0$

either $m^2 = -9 \Rightarrow m_{1,2} = \mp 3i$

or $m^2 = -1 \Rightarrow m_{3,4} = \mp i$

$y_h = e^{0.x}[C_1 \sin 3x + C_2 \cos 3x] + e^{0.x}[C_3 \sin x + C_4 \cos x]$

$\Rightarrow y(x) = y_h = C_1 \sin 3x + C_2 \cos 3x + C_3 \sin x + C_4 \cos x$

③ في حالة وجود (y) وليس هناك تناظر بالاسس لإستعمال الدستور نستخدم طريقة التجزئة او التجربة والخطأ (Trail and error) .

ملاحظة ← لغرض إستعمال التجربة والخطأ يتم فرض قيمة (m) التي تصفر الطرف الايمن وكما يلي ←

ex $m^3 + m^2 + 2m + 2 = 0$ ∓ 1, ∓ 2 (1×2)

$m = \mp 1$ $(m+1)$

$* \; m = 1 \Rightarrow 1 + 1 + 2 \times 1 + 2 \neq 0$

$m = -1 \Rightarrow -1 + 1 - 2 + 2 = 0 \Rightarrow \therefore$ root $= -1$ $(m+1)$

$$\begin{array}{r|l} & m^2 + 2 \\ \hline m+1 & m^3 + m^2 + 2m + 2 \\ & \mp m^3 \mp m^2 \\ \hline & 2m + 2 \\ & \mp 2m \mp 2 \\ \hline & \end{array}$$

بالطرح

بالطرح

$\frac{m^3}{m} = m^2$

$m^2 = 2$

$\frac{2m}{m} = 2$

① نفرض قيمة m اصغر يجوز

② القسمة على الحد الاول

③ الضرب

④ الطرح مع ملاحظة الإشارات ونزول الباقي .

$(m+1)(m^2+2) = 0$

$m_1 = -1$ $\quad$ $m_2 = m_3 = \mp\sqrt{-2} = \mp\sqrt{2}\, i$

$y(x) = y(h) = C_1 e^{-x} + e^{0.x}[C_2 \sin\sqrt{2}x + C_3 \cos\sqrt{2}\,x]$

$(m^3+m^2)+(2m+2)=0 \Rightarrow m^2(m+1)+2(m+1)=0$ طريقة التجزئة

$\Rightarrow (m+1)(m^2+2)=0$

either $m_1=-1$ or $m_2=m_3=\mp\sqrt{-2}=\mp\sqrt{2}\,i$

b) Non-Homogenous linear D.E with Constant Coeff.:-

→ The standard form:

$$a\frac{d^n y}{dx^n}+b\frac{d^{n-1}y}{dx^{n-1}}+\dots = \text{Constant or } f(x)$$

and the solution $\Rightarrow y(x)=y_c+y_p$

y_c, y_p → U.C, V.P

U.C → $e^{m\cdot x}$, sin or cos or both, polynomail

$x^2+4=0$

$x_{1,2}=\mp$

$x=\mp 2$

b-1) Undeterminate Coeff. method (U.C):- y_c y_p

Ex-① Solve $\frac{d^5y}{dx^5}-\frac{d^3y}{dy^3}=3x^2+2e^x+10x-8$

Sol. To find $y_c=y^{V}-y'''=0 \Rightarrow m^5-m^3=0$

$\Rightarrow m^3(m^2-1)=0$ → $m^2=1 \to m_{4,5}=\mp\sqrt{1}$

either or $m_4=1$

$m_3=0$ $m_5=-1$

$m_1=m_2=m_3=0$

$P=3-1=2$

$y_c=C_1e^{x}+C_2e^{-x}+e^{0\cdot x}[C_3x^2+C_4x+C_5]$

$$\boxed{y_c=C_1e^{x}+C_2e^{-x}+C_3x^2+C_4x+C_5}$$

$3x^2 + 2e^x + 10x - 8$ poly

To find y_p : $f(x)_1 = 2e^x \Rightarrow y_{p1} = \boxed{Ae^x}\, x$

$f(x)_2 = 3x^2 + 10x + 8$

ثلاث حدود من yp تشبه $\quad y_{p2} = (Bx^2 - Cx + D) * x^3$ $\quad + Bx^3 - Cx^2 + Dx$

ثلاث حدود من yh $\quad = Bx^5 - Cx^4 + Dx^3$ $\quad Bx^4 - Cx^3 + Dx^2$ $\quad x\ x^3$

$$\boxed{\rightarrow y_p = y_{p1} + y_{p2} = Ae^x \cdot x + Bx^5 + Cx^4 + Dx^3}$$

To evaluate A, B, C, D we find

$3x \rightarrow 3$

$y'_p = Axe^x + Ae^x + 5Bx^4 + 4Cx^3 + 3Dx^2$

$y''_p = Axe^x + 2Ae^x + 20Bx^3 + 12Cx^2 + 6Dx$

$\rightarrow y'''_p = Axe^x + 3Ae^x + 60Bx^2 + 24Cx + 6D$

$y''''_p = Axe^x + 4Ae^x + 120Bx + 24C$

$\rightarrow y''''' = Axe^x + 5Ae^x + 120B$

sub y''''_p and y'''_p in D.E $\quad d^3y/dx^3$

d^5y/dx^5

$$(Axe^x + 5Ae^x + 120B) - (Axe^x - 3Ae^x - 60Bx^2 - 24Cx - 6D)$$

$$= 3x^2 - 2e^x + 10x - 8$$

- Coeff. of $e^x \Rightarrow 5A - 3A = 2 \Rightarrow A = 1$ ✓
- 〃 〃 $x^2 \Rightarrow -60B = 3 \Rightarrow B = \frac{-1}{20}$ ✓
- 〃 〃 $x \Rightarrow -24C = 10 \Rightarrow C = \frac{-5}{12}$ ✓

- constant $\Rightarrow 120\cancel{B} - 6D = -8$

$120 * \frac{-1}{20} - 6D = -8$

$D = \frac{1}{3}$

$$y_p = xe^x - \frac{x^5}{20} - \frac{5x^4}{12} + \frac{x^3}{3}, \quad y(x) = y_h + y_p$$

✓ $$y(x) = C_1e^x + C_2e^{-x} + C_3x^2 + C_4x + C_5 + xe^x - \frac{x^5}{20} - \frac{5x^4}{12} + \frac{x^3}{3}$$

4) **Ex (2)** Solve $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 6y = e^{3x}\sinh(3x)$

$\because f(x) = e^{3x}\sinh(3x) = e^{3x}\left(\frac{e^{3x} - e^{-3x}}{2}\right) = \frac{e^{6x}}{2} - \frac{1}{2}$

To find yp

$m^3 + 2m^2 + 3m + 6 = 0$

$m(m^2+3) + 2(m^2+3) = 0 \Rightarrow (m^2+3)(m+2) = 0$

either $m_1 = -2$ or $m_2, m_3 = \mp\sqrt{3}i$

$y_h = C_1 e^{-2x} + e^{0x}[C_2 \cos\sqrt{3}x + C_3 \sin\sqrt{3}x]$

$= C_1 e^{-2x} + C_2\cos\sqrt{3}x + C_3 \sin\sqrt{3}x$

To find yp $\because f(x)_1 = \frac{e^{6x}}{2} \Rightarrow y_{p1} = Ae^{6x}$

$f(x)_2 = \frac{-1}{2} \Rightarrow y_{p2} = B$

$y_p = y_{p1} + y_{p2} = Ae^{6x} + B$

To evaluate A and B, we find

$y'_p = 6Ae^{6x}$, $y''_p = 36Ae^{6x}$, $y'''_p = 216Ae^{6x}$

Subs. in O.D.E $\Rightarrow 216Ae^{6x} + 72Ae^{6x} + 18Ae^{6x} + Ae^{6x} + 6B = \frac{e^{6x}}{2} - \frac{1}{2}$

Coeff. of $e^{6x} = 216A + 72A + 6A = \frac{1}{2} \Rightarrow A = 1$

" " Constant $= 6B = -\frac{1}{2} \Rightarrow B = \frac{-1}{3}$

$y(x) = y_h + y_p$

EX (3) Solve $\underbrace{\left(\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2}\right)}_{y_c} = \underbrace{(10\sin x - 3\cos x + 10)}_{y_p}$

Sol: * To find $y_c \Rightarrow y''' + y'' = 0$

$m^4 + m^2 = 0 \Rightarrow m^2(m^2+1) = 0$

either $m^2 = 0$, $m_1 = m_2 = 0$

$m^2 = -1$, $m_3, m_4 = \mp i$, $\alpha = 0$, $B = 1$

$$y_h = e^{0\cdot x}(C_1 x + C_2) + e^{0\cdot x}(C_3 \cos x + C_4 \sin x)$$

$$\boxed{y_c = C_1 x + C_2 + C_3 \cos x + C_4 \sin x}$$

To find $y_p \Rightarrow$

$f(x)_1 = 10 \sin x - 3 \cos x$

$y_{p_1} = (A \cos x - B \sin x) * x$

$y_{p_1} = A x \cos x + B x \sin x$

$f(x)_2 = 10 \Rightarrow y_{p_2} = C] * x^2 \Rightarrow y_{p_2} = C x^2$

$y_p = y_{p_1} + y_{p_2} = x A \cos x + x B \sin x + C x^2$

To evaluate A, B, C we find

$y'_p =$

$y''_p =$

$y'''_p =$

$y''''_p =$

Then $y(x) = y_h + y_p$

4- Application of Higher order D.E

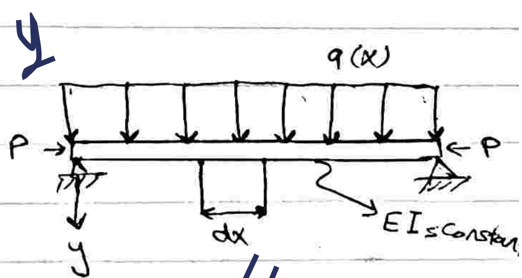
Beam – Column D.E

$\sum F_y = 0 \; + \downarrow$

$-V + (V + dV) + q(x)\,dx = 0$

$-\cancel{V} + \cancel{V} + dV + q(x)\,dx = 0$

$q(x) = -\dfrac{dV}{dx}$ ------ (1)



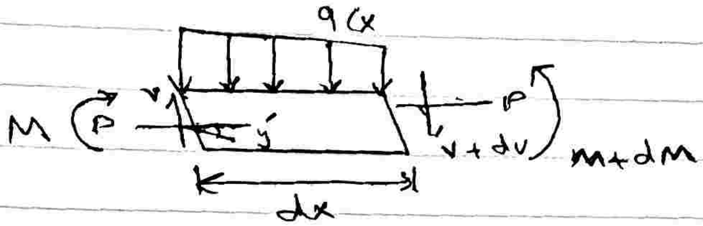
$y, y'' = 0$

$\sum M = 0$

$M - (M + dM) + (V + dV)\,dx + q(x)\cdot dx \cdot \dfrac{dx}{2} + P\dfrac{dy}{dx}\cdot dx = 0$

All of the term of the 2nd order will be neglected & the equation will be divided by (dx) to get

$\cancel{M} - \cancel{M} + dM + V\,dx + \underbrace{\left(dV\,dx + q(x)\dfrac{dx^2}{2}\right)}_{\text{neglect}} + P\cdot dy = 0] \div dx$



All of the term of the 2nd order will be neglected and the equation will be divided by (dx) to get

$$\cancel{M} - \cancel{M} + dM + V\,dx + dv\,dx + q(x)\frac{dx^2}{2} + P.dy = 0\} \div dx$$

$$V = \frac{dM}{dx} - P.\frac{dy}{dx} \Longrightarrow \frac{dv}{dx} = \frac{d^2M}{dx^2} - P\frac{d^2y}{dx^2}$$

We have from the derivative of O.D.E. of beam deflection

$$EI\frac{d^2y}{dx^2} = -M$$

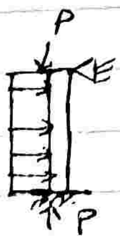
Then :-

$$-EI\frac{d^4y}{dx^4} - P\frac{d^2y}{dx^2} = \frac{dv}{dx} \quad \text{------} (2)$$

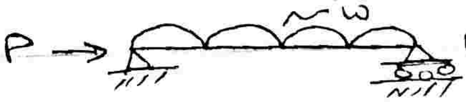
sub. Eq. (1) into Eq. (2) we get :-

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} = [q(x)]$$

ملاحظة: تستعمل معادلة الـ load في تساوي الحل المنتشر في الحالات التالية :-



(2)



(1)

Axial force or (beam - Column)

Undeterminate

(3)

Ex ① Find the max. deflection of the clamped beam.

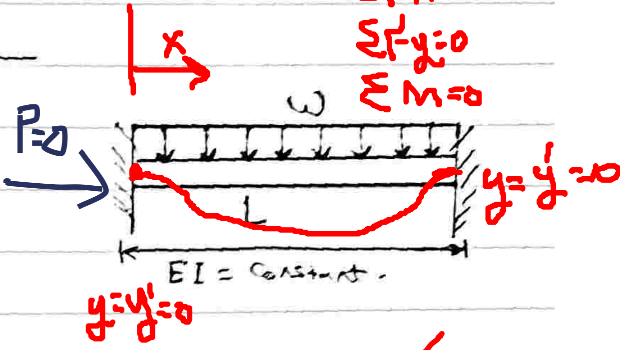
Sol.

$\because$ undeterminate beam so using :—

$$\boxed{EI\frac{d^4y}{dx^4}+P\cdot\frac{d^2y}{dx^2}=q(x)}$$

and $P=0$, $q(x)=w$

$$\therefore \boxed{EI\frac{d^4y}{dx^4}=w}$$



$\sum F_x=0$
$\sum F_y=0$
$\sum M=0$

$x=0$

$$EI\int\frac{d^4y}{dx^4}=\int w$$

$$EI\int\frac{d^3y}{dx^3}=\int wx+A$$

$$EI\int\frac{d^2y}{dx^2}=\int\frac{wx^2}{2}+Ax+B$$

$$EI\int\frac{dy}{dx}=\int\frac{wx^3}{6}+\frac{Ax^2}{2}+Bx+C$$

$$\boxed{EI\,y(x)=\frac{w}{24}x^4+\frac{Ax^3}{6}+\frac{Bx^2}{2}+Cx+D}$$

(deflection) معادلة الهطول

A, B, C, D

B.C ① $x=0,\ y=0 \Rightarrow D=0$

B.C ② $x=0,\ y'=0 \Rightarrow C=0$

B.C ③ $x=l,\ y=0 \quad 0=\frac{wl^4}{24}+\frac{A}{6}l^3+\frac{B}{2}l^2 \quad \div L^2\times 2$

$$B=-\frac{w}{12}L^2-\frac{AL}{3} \quad \text{---} ①$$

B.C ④ $x=l,\ y'=0 \quad 0=\frac{wl^3}{6}+\frac{A}{2}l^2+Bl \quad \div L$

$$B+A\frac{l}{2}=-\frac{wl^2}{6} \quad \text{---} ②$$

Sub. eq. ① into eq. ②

$$-\frac{w}{12}L^2-\frac{AL}{3}+A\frac{L}{2}=-\frac{wL^2}{6}\Longrightarrow\frac{L}{6}A=-\frac{wL^2}{6}+\frac{wL^2}{12}$$

$$\Longrightarrow\frac{L}{6}A=-\frac{w}{12}L^2\Longrightarrow\boxed{A=\frac{w}{2}L}$$

$$\therefore B=\frac{-w}{12}L^2-\left(-\frac{w}{2}L\right)\cdot\frac{L}{3}=\boxed{\frac{w}{12}L^2}$$

Due to Symmetry the max. deflection of the beam should be at $\left(x=\frac{L}{2}\right)$

$$EI\,y(x)=\frac{w}{24}x^4-\frac{wx^3L}{12}+\frac{wx^2L^2}{24}$$ (deflection) معادلة الـ

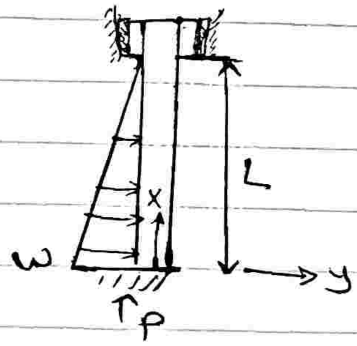
$$y(x)=\frac{w}{24EI}\left(x^4-2Lx^3+L^2x^2\right)$$

$$y_{max}=y\left(\frac{l}{2}\right)=\frac{w}{24EI}\left[\left(\frac{l}{2}\right)^4-2l\left(\frac{l}{2}\right)^3+l^2\left(\frac{l}{2}\right)^2\right]=\boxed{\frac{wl^4}{38EI}}$$

Ex.② Find the deflection equation of the clamped column.

Sol. The O.D.E that satisfied the deflection & buckling of indeterminate stru. is :-

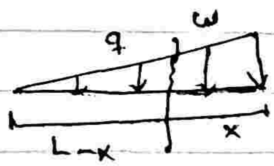
$$EI\frac{d^4y}{dx^4}+P\frac{d^2y}{dx^2}=q(x)$$



Where $q(x)=\frac{w}{L}(L-x)$, the O.DE. will be :-

$$EI\frac{d^4y}{dx^4}+P\frac{d^2y}{dx^2}=\frac{w}{L}(L-x)\,]\div EI$$

$$\frac{d^4y}{dx^4}+\frac{P}{EI}\frac{d^2y}{dx^2}=\frac{w}{L\cdot EI}(L-x)$$



- This is a higher non homogenous O.D.E
- The characteristic Eq. is :- (to find y_h)

$$m^4+\frac{P}{EI}m^2=0\Longrightarrow m^2\left(m^2+\frac{P}{EI}\right)=0$$

Either :- $m^2=0$, $m_1, m_2=0$ (or) $\left(m^2+\frac{P}{EI}\right)=0$, $m_3, m_4=\mp\sqrt{\frac{P}{EI}}\,i$ of $(\alpha+\beta i)$

$$\rightarrow y_h = e^{0\cdot x}\left[C_1 x + C_2\right] + e^{0\cdot x}\left[C_3 \cos\sqrt{\frac{P}{EI}}\,x + C_4 \sin\sqrt{\frac{P}{EI}}\,x\right]$$

$$\rightarrow y_h = C_1 x + C_2 + C_3 \cos\sqrt{\frac{P}{EI}}\,x + C_4 \sin\sqrt{\frac{P}{EI}}\,x$$

To find y_p :- $f(x) = \frac{w}{EIL}(L-x) = \frac{w}{EI} - \frac{wx}{L\cdot EI}$

$\therefore y_p = [Ax + B] * x^2$ [لوجود تشابه مع y_h]

$y_p = Ax^3 + Bx^2$

$y'_p = 3Ax^2 + 2Bx$, $y''_p = 6Ax + 2B$, $y''' = 6A$, $y^{IV} = 0$

sub. the derivatives in the O.D.E to get $\leftarrow$

$$0 + \frac{P}{EI}(6Ax + 2B) = \frac{w}{EI} - \frac{w}{EI\cdot L}\cdot x$$

$$\frac{6P}{EI}Ax + \frac{2P}{EI}B = \frac{w}{EI} - \frac{w}{L\cdot EI}x$$

Coeff. of x $\Rightarrow \frac{6PA}{EI} = -\frac{w}{L\cdot EI} \Rightarrow A = \frac{-w}{6PL}$

Coeff. of Constant $\Rightarrow \frac{2P}{EI}B = \frac{w}{EI} \Rightarrow B = \frac{w}{2P}$

$$\therefore y_p = \frac{-w}{6PL}x^3 + \frac{w}{2P}x^2$$

$\therefore y(x) = y_h + y_p$

$$y(x) = C_1 x + C_2 + C_3 \cos\sqrt{\frac{P}{EI}}\,x + C_4 \sin\sqrt{\frac{P}{EI}}\,x - \frac{w}{6PL}x^3 + \frac{w}{2P}x^2$$

B.C ① $\begin{cases} x = 0 \\ y = 0 \end{cases}$ $C_2 + C_3 = 0$ ---- ① $\boxed{C_2 = -C_3}$

B.C ② $\begin{cases} x = 0 \\ y' = 0 \end{cases}$ $\frac{dy}{dx} = C_1 + 0 - C_3\sqrt{\frac{P}{EI}}\cdot\sin\sqrt{\frac{P}{EI}}\,x + C_4\sqrt{\frac{P}{EI}}\cdot\cos\sqrt{\frac{P}{EI}}\,x - \frac{wx^2}{2PL} + \frac{wx}{P}$

$0 + 0 - 0 + C_4\sqrt{\frac{P}{EI}} \Rightarrow C_1 = -C_4\sqrt{\frac{P}{EI}}$ ---- ②

B.C ③ $\begin{cases} x = l \\ y = 0 \end{cases}$ $0 = C_1 l + C_2 + C_3\cos\sqrt{\frac{P}{EI}}\,l + C_4 \sin\sqrt{\frac{P}{EI}}\,l - \frac{wl^2}{6P} + \frac{wl^2}{2P}$ ---- ③

B.C ④ $\begin{cases} x = l \\ y' = 0 \end{cases}$ $0 = C_1 - C_3\sqrt{\frac{P}{EI}}\cdot\sin\sqrt{\frac{P}{EI}}\,L + C_4\sqrt{\frac{P}{EI}}\cos\sqrt{\frac{P}{EI}}\cdot L - \frac{wL}{2P} + \frac{wL}{P}$ ---- ④

Sub. Eq. ① and ② into Eq ③ and ④

$$-\frac{wL^2}{6P} + \frac{wL^2}{2P}$$

$$0 = -\sqrt{\frac{P}{EI}}\, C_4 . L - C_3 + C_3 \cos\sqrt{\frac{P}{EI}}\, L + C_4 \sin\sqrt{\frac{P}{EI}}\, L$$

$$\Rightarrow C_3\left(\cos\sqrt{\frac{P}{EI}}\, L - 1\right) + C_4\left(\sin\sqrt{\frac{P}{EI}}\, L - \sqrt{\frac{P}{EI}}\, L\right) + \frac{wL}{2P} = 0 \quad \text{—③}$$

$$0 = -\sqrt{\frac{P}{EI}} \cdot C_4 . L - C_3\sqrt{\frac{P}{EI}} \sin\sqrt{\frac{P}{EI}}\, L + C_4\sqrt{\frac{P}{EI}} \cos\sqrt{\frac{P}{EI}}\, L - \frac{wL}{2P} + \frac{wL}{P}$$

$$\Rightarrow -C_3\sqrt{\frac{P}{EI}} \cdot \sin\sqrt{\frac{P}{EI}}\, L + C_4\sqrt{\frac{P}{EI}}\left[\cos\sqrt{\frac{P}{EI}}\, L - 1\right] + \frac{wL}{2P} = 0 \quad \text{—④}$$

① $\Rightarrow$ Eq. ③] $* \sqrt{\frac{P}{EI}} \sin\sqrt{\frac{P}{EI}}\, L + \left[\cos\sqrt{\frac{P}{EI}}\, L - 1\right] *$ Eq — ④

② $\Rightarrow$

$$C_4 = \frac{-\frac{wL^2}{3P}\sqrt{\frac{P}{EI}} \sin\sqrt{\frac{P}{EI}}\, L + \frac{wL}{2P}\left(\cos\sqrt{\frac{P}{EI}}\, L - 1\right)}{\left(\sqrt{\frac{P}{EI}} \sin\sqrt{\frac{P}{EI}}\, L\right)\left(\sin\sqrt{\frac{P}{EI}}\, L - \sqrt{\frac{P}{EI}}\, L\right) + \sqrt{\frac{P}{EI}}\left(\cos\sqrt{\frac{P}{EI}}\, L - 1\right)}$$

③ $\Rightarrow \left(\sin\sqrt{\frac{P}{EI}}\, L - \sqrt{\frac{P}{EI}}\right) *$ Eq ④ $- \sqrt{\frac{P}{EI}}\left(\cos\sqrt{\frac{P}{EI}}\, L - 1\right) *$ Eq. ③

④ $\Rightarrow$

$$C_3 = \frac{-\left(\frac{wL^2}{3P}\sqrt{\frac{P}{EI}}\left[\cos\sqrt{\frac{P}{EI}} - \sqrt{\frac{P}{EI}}\right] + \frac{wL}{2P}\left[\sin\sqrt{\frac{P}{EI}}\, L - \sqrt{\frac{P}{EI}}\right]\right)}{\left(\sqrt{\frac{P}{EI}} \sin\sqrt{\frac{P}{EI}}\, L \sin\sqrt{\frac{P}{EI}} - \sqrt{\frac{P}{EI}}\, L\right) + \sqrt{\frac{P}{EI}}\left(\cos\sqrt{\frac{P}{EI}} - 1\right)^2}$$

$$\Rightarrow C_1 = -C_3$$

$$\Rightarrow C_2 = \sqrt{\frac{P}{EI}}\, C_4$$

Deflection of beams:

The common basis of the available analytical methods, used to determine the deflection of beams, is the differential equation that relates the deflection to the bending moment:

$$EI\ddot{y} = -M(x) \qquad (1)$$

E : modulus of elasticity

I : moment of inertia

y : deflection

And by derivation we get:

$$EI\dddot{y} = -V(x) \qquad (2)$$

$$EIy^{IV} = -w(x) \qquad (3)$$

M(x) : bending moment at point located at x from left end.

V(x) : shear force at point located at x from left side

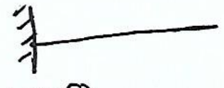
w(x) : uniformly distributed load at x from left end

Sign convention:

y: $+\downarrow$ $\uparrow-$ $\quad$ $w(x)$: $-\downarrow$ $\uparrow+$ $\quad$ $M(x)$: $-$ $+$

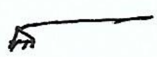
$V(x)$: $+$ $-$

Boundary conditions:



$y=0$

$\theta = \dot{y} = 0$



$y=0$

$M=0 \Rightarrow \ddot{y}=0$

free end

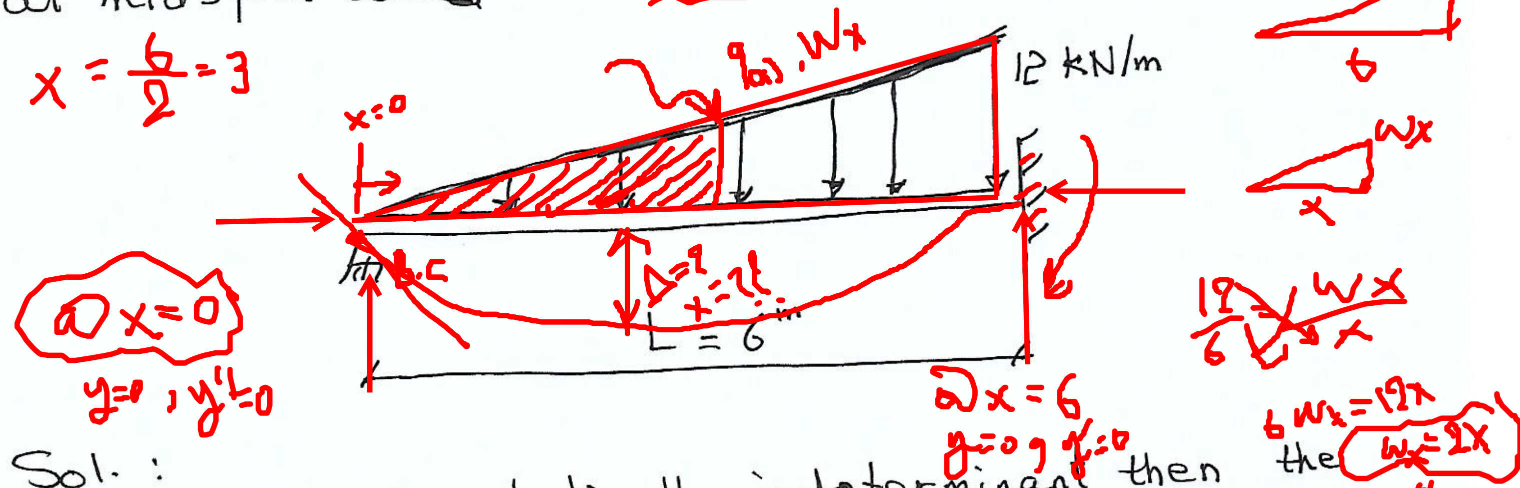
$M=0 \Rightarrow \ddot{y}=0$

$V=0 \Rightarrow \dddot{y}=0$

Note 1:
If $w(x)$ can't be defined as a single function for the whole length of the beams, then the previous equations shall be applied for each part with different defintion.

Note 2:
If the $M(x)$ can't be calculated statically, then equation 3 shall be applied.

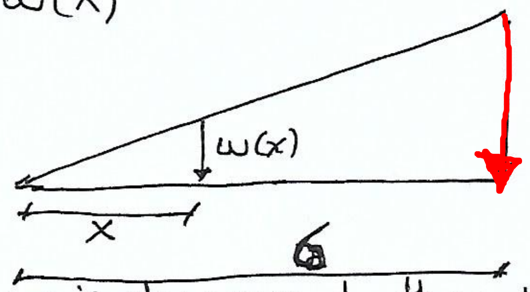
Exa.:
For the beam shown below, find shear and moment at midspan and the max. deflection.



Sol.:
As the beam is statically indeterminant then equ. 3 shall be applied:

$$EIy^{IV} = -w(x)$$

· Finding $w(x)$:



$$\frac{w(x)}{x} = \frac{12}{6}$$

$\therefore \boxed{w(x) = 2X}$, as $w(x)$ is downward then $w(X) = -2X$

$$EIy^{IV} = -(-2x) = 2X$$

$$EIy''' = x^2 + c_1 \quad \text{--- (1)}$$

$$EIy'' = \frac{x^3}{3} + c_1 x + c_2 \quad \text{--- (2)}$$

$$EIy' = \frac{x^4}{12} + c_1\frac{x^2}{2} + c_2 x + c_3 \quad \text{--- (3)}$$

$$EIy = \frac{x^5}{60} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3 x + c_4 \quad \text{--- (4)}$$

Boundary conditions:

$y = 0$ at $x = 0$ —— (5)

$y'' = 0$ at $x = 0$ —— (6)

$y = 0$ at $x = 6$ —— (7)

$y' = 0$ at $x = 6$ —— (8)

Then

$0 = 0 + 0 + 0 + 0 + C_4 \Rightarrow \boxed{C_4 = 0}$ from equ. 5

$0 = 0 + 0 + C_2 \Rightarrow \boxed{C_2 = 0}$ from equ. 6

$0 = \frac{6^5}{60} + C_1 \frac{6^3}{6} + 0 + 6C_3 + 0$ from equ. 7

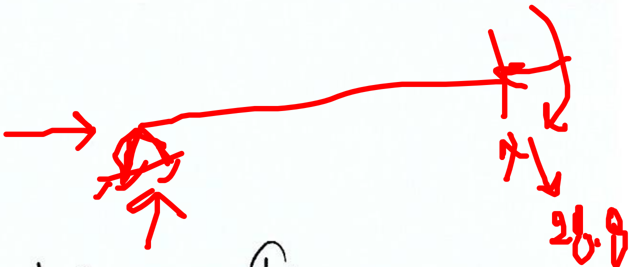
$\therefore 36C_1 + 6C_3 = -129.6$ —— (9)

$0 = \frac{6^4}{12} + 18C_1 + 0 + C_3$ from equ. 8

$18C_1 + C_3 = -108$ —— (10)

Equ. (9) and Equ. (10) solved to find:

$$\boxed{\begin{aligned} C_1 &= -7.2 \\ C_3 &= 21.6 \end{aligned}}$$



To find shear at midspan:

$x = 3$ is substituted in equ. (1)

$EI\,y''' = -V(x) = x^2 + C_1$

$\therefore V(3) = -\left[3^2 + (-7.2)\right] = -1.8 \text{ kN}$

28.0

To find B.M at midspan:

$x = 3$ is substituted in equ. (2)

$EI\,y'' = -M(x) = \frac{x^3}{3} - 7.2x + 0$

$\therefore M(3) = -\left[\frac{27}{3} - 7.2(3)\right] = 12.6 \text{ m-kN}$

And to find max. deflection:

$$y' = 0$$

$$\Longrightarrow \frac{x^4}{12} - 7.2\,\frac{x^2}{2} + 21.6 = 0 \quad \text{——} (11)$$

By solving equ. 11 we find x at which the max. deflection occurs.

Then the value of x will substituted in eq. (4) to find the max. deflection.

B. Second and Higher order Differential Equation with Variable Coefficient:

(1) Special Case (Cauchy Equation or Euller Equation:

ملاحظة : الشرط لاستخدام هذه الطريقة هو يجب ان تكون درجة أس المعامل مساوية لدرجة المشتقة المضروب بها.

Second order:

$$A x^2 y'' + B x y' + C y = 0$$

Procedure:

طريقة الحل

1. Assume $y = x^m$ or $y = (x-a)^m$ and find its required derivatives. Then, substitute it in the original differential equation.

2. The resulted equation is divided by x^m or $(x-a)^m$.

3. From the resulted equation the values of m are found to determine the general solution of the differential equation as follow:

a. In the case of different real roots

$$y_h = c_1 x^{m_1} + c_2 x^{m_2}$$

b. In the case of imaginary roots ($m_{1,2} = \alpha \pm i\beta$)

$$y_h = x^{\alpha} (c_1 \sin \beta \ln x + c_2 \cos \beta \ln x)$$

c. If the real roots are coincided

$$y_h = c_1 x^m + c_2 (\ln x) x^m$$

Example 1:

Find the general solution of the following differential equation

$$(x^2 - 2x + 1) y'' + (5x - 5) y' + 6y = 0$$

$(x-1)^2$

Solution:

$$(x-1)^2 y'' + 5(x-1) y' + 6y = 0$$

Let: $y = (x-1)^m$ assume

$y' = m[(x-1)^{m-1}]$

$y'' = m(m-1)(x-1)^{m-2}$

$$(x-1)^2 [m(m-1)(x-1)^{m-2}] + 5(x-1)[m(x-1)^{m-1}] + 6(x-1)^m = 0$$

$$(m^2 - m)\cancel{(x-1)^m} + 5m\cancel{(x-1)^m} + 6\cancel{(x-1)^m} = 0 \quad \div (x-1)^m$$

$$m^2 - m + 5m + 6 = 0$$

$$m^2 + 4m + 6 = 0$$

$Ax^2 + Bx + c = 0$

$x_1, x_2 = \dfrac{-B \mp \sqrt{B^2 - 4Ac}}{2A}$

$$m_{1,2} = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(6)}}{2(1)} = -2 \pm i \frac{\sqrt{8}}{2}$$

$= \sqrt{2}$

$\sqrt{-8} = \sqrt{8} \cdot \sqrt{-1}$

$$= -2 \pm i\sqrt{2}$$

$\frac{3+5}{2} = \frac{3}{2} + \frac{5}{2}$

$\frac{2}{3+5} \neq \frac{2}{3} + \frac{2}{5}$

$\therefore \alpha = -2$ and $\beta = \sqrt{2}$

$y_p = 0$

$$y_c = y_h = (x-1)^{-2} \left[c_1 \sin\sqrt{2}\, \ln|x-1| + c_2 \cos\sqrt{2}\, \ln|x-1| \right]$$

Example 2:

Find the general solution of the following differential equation.

$$(2x^2 - 4x + 2)\,y'' - (4 - 4x)\,y' + 5y = 0$$

$(x-1)$

Solution:

$$2(x^2 - 2x + 1)\,y'' - 4(1-x)\,y' + 5y = 0$$

$$2(x-1)^2\,y'' + 4(x-1)\,y' + 5y = 0$$

Let: $y = (x-1)^m$

$y' = m(x-1)^{m-1}$

$y'' = m(m-1)(x-1)^{m-2}$

$$2(x-1)^2\left[m(m-1)(x-1)^{m-2}\right] + 4(x-1)\left[m(x-1)^{m-1}\right] + 5(x-1)^m = 0$$

$$2m(m-1)\cancel{(x-1)^m} + 4m\cancel{(x-1)^m} + 5\cancel{(x-1)^m} = 0 \qquad \div (x-1)^m$$

$$2m^2 - 2m + 4m + 5 = 0$$

A B C

$$2m^2 + 2m + 5 = 0$$

$$\therefore m_{1,2} = \frac{-2 \pm \sqrt{(2)^2 - 4(2)(5)}}{2(2)} = \frac{-2 \pm \sqrt{4(1-10)}}{4}$$

$$= \frac{-2 \pm i\,2\sqrt{9}}{4}$$

α β

$$m_{1,2} = -\frac{1}{2} \pm i\,\frac{3}{2} \implies \alpha = -\frac{1}{2} \text{ and } \beta = \frac{3}{2}$$

$$\therefore y_h = (x-1)^{-1/2}\left[c_1 \sin\frac{3}{2}\ln|x-1| + c_2 \cos\frac{3}{2}\ln|x-1|\right]$$

Higher order:

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = 0$$

ملاحظة : لايجاد y_h نستخدم نفس الطريقة السابقة في حل المعادلات التفاضلية من الدرجة الثانية.

$y_h = c_1 x^{m_1} + c_2 x^{m_2} + c_3 x^{m_3} + \cdots$ حقيقية مختلفة

خيالية ← الحل يعتمد على نوع الجذور

$y_h = x^m (c_1 + c_2 \ln x + c_3 \ln^2 x + \cdots)$ حقيقية متشابهة

Example:

Find the general solution of the following differential equation.

$$\overset{A}{x^3} y''' - \overset{B}{3x^2} y'' + \overset{C}{6x} y' - \overset{D}{6} y = 0$$

x^{-1}

x

Solution:

Let $y = x^m$

$y' = m x^{m-1}$

$y'' = m(m-1) x^{m-2}$

$y''' = m(m-1)(m-2) x^{m-3}$ → $(m^2 - m)(m-2) x^{m-3}$

$(m^3 - 2m^2 - m^2 + 2m)$

$x^4 \rightarrow 4x^3$

$$x^3 (m^3 - 3m^2 + 2m) x^{m-3} - 3x^2 (m^2 - m) x^{m-2} + 6x\, m x^{m-1} - 6x^m = 0$$

$$(m^3 - 3m^2 + 2m)\cancel{x^m} - (3m^2 - 3m)\cancel{x^m} + 6m\cancel{x^m} - 6\cancel{x^m} = 0 \quad \div x^m$$

$$m^3 - 3m^2 + 2m - 3m^2 + 3m + 6m - 6 = 0$$

$$m^3 - 6m^2 + 11m - 6 = 0$$

m_1, m_2, m_3

نبدأ الحل بالتجربة والخطأ وذلك بفرض قيمة لـ m فاذا تحققت فأنها تمثل الحل الاول .

Let $m = 1 \Longrightarrow (1)^3 - 6(1)^2 + 11(1) - 6 = 0$

$0 = 0 \quad \therefore$ o.K

$\therefore m_1 = 1$ is the first solution.

$m^3 = m^2$

لايجاد الحل الثاني والثالث يكون بالقسمة الطويلة .

$m-1$

① نقسم حد واحد
② نضرب
③ نطرح
④ نزل ابناء

$$
\begin{array}{r|l}
 & m^2 - 5m + 6 \\
\hline
(m-1) & m^3 - 6m^2 + 11m - 6 \\
\text{بالطرح} & \mp m^3 \pm m^2 \\
\hline
 & -5m^2 + 11m - 6 \\
 & \pm 5m^2 \mp 5m \\
\hline
 & 6m - 6 \\
 & \mp 6m \pm 6 \\
\hline
 & 0 + 0
\end{array}
$$

$m_1 = 1$

$m_2 = 2$

$m_3 = 3$

$m^2 - 5m + 6 = 0$

$(m-2)(m-3) = 0$

$m - 2 = 0 \Longrightarrow m_2 = 2$

$m - 3 = 0 \Longrightarrow m_3 = 3$

$y_c = y_h = c_1 x + c_2 x^2 + c_3 x^3$

$y_p = 0$

7.3 Buckling of Columns

Buckling of columns is considered as a phenomenon of unstability. Its occurrence depends on the slenderness of the column (λ) which can be defined as the ratio of column effective length to its radius of gyration as follows:

$$\lambda = \frac{L_e}{r} \qquad (7\text{-}24)$$

Where,

L_e = column effective length = KL, in which K = effective length factor

r = radius of gyration = $\sqrt{\frac{I}{A}}$ in which I = moment of inertia of the column section, and A = the column cross-sectional area.

The value of effective length factor depends on the method of column ends supports. Figure (7-13), which was quoted from American Institute for Steel Construction AISC 360-05 Specification Table C-C2.2, shows this value for different column end conditions

TABLE C-C2.2
Approximate Values of Effective Length Factor, *K*

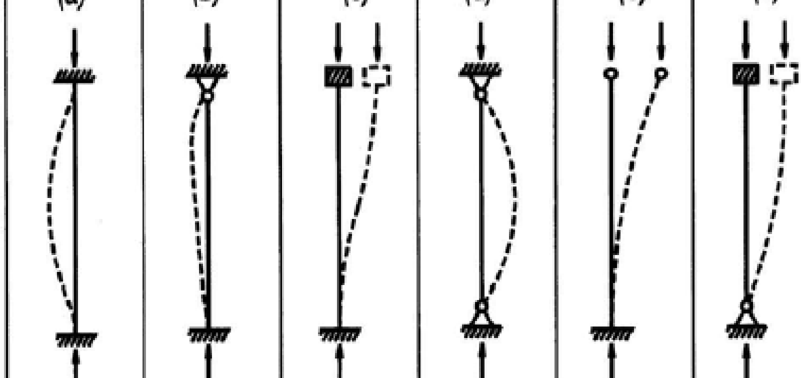
Buckled shape of column is shown by dashed line.	(a)	(b)	(c)	(d)	(e)	(f)
						
Theoretical *K* value	0.5	0.7	1.0	1.0	2.0	2.0
Recommended design value when ideal conditions are approximated	0.65	0.80	1.2	1.0	2.10	2.0
End condition code	*Rotation fixed and translation fixed* *Rotation free and translation fixed* *Rotation fixed and translation free* *Rotation free and translation free*					

Figure (7-13) Values of Effective length factor for different column end conditions

7.3.1 Pined-pined column

Figure (7-14) shows a pined-pined axially-loaded member (column) subject to the axial load (P) in its longitudinal direction x and buckled in y-direction with a maximum value of (δ) at column mid-length. The governing differential equation for this buckling can be derived from the equation of moment that induced by existence the force (P) and the buckling displacement (y). So that:

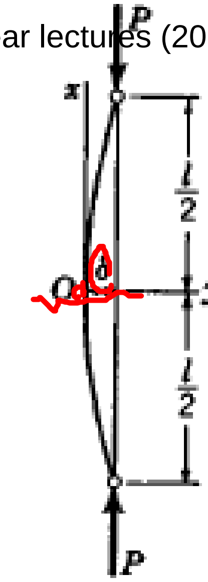


Figure (7-14) Pined-pined column

$M(x) = -Py$

$EIy" = -Py$

$$y'' + \frac{P}{EI} y = 0$$

yc yp

$Let\ \frac{P}{EI} = k^2, then:$

$$y'' + k^2 y = 0 \qquad (7-25)$$

Equation (7-25) is classified as: Homogeneous 2nd Order D.E. It can be solved by the method of undetermined coefficients using the characteristic equation as follows:

$$m^2 + k^2 = 0 \rightarrow m^2 = -k^2 \rightarrow m_{1,2} = \mp ki \ (Where\ i = imajinary\ root = \sqrt{-1}$$

Accordingly the solution is:

$$y(x) = Asinkx + Bcoskx$$

To find the arbitrary constants A and B apply the following boundary conditions:

B.C

(a) $y(0) = 0 \rightarrow B = 0$

(b) $y(l) = 0 \rightarrow 0 = Asinkl \rightarrow sinkl = 0 \rightarrow kl = n\pi$

By squaring both sides:

$$k^2 l^2 = n^2\pi^2 \rightarrow \frac{P}{EI} l^2 = n^2\pi^2 \rightarrow \frac{P}{EI} = \frac{n^2\pi^2}{l^2}$$

$$\boxed{P_{cr} = \frac{\pi^2 EI}{l^2}}$$

P_{cr} = is the critical axial load at which buckling is occurred. It is also referred as *Euler's Load*.

n = represent number of ripples in the buckled shape of the column.

l = effective length of the column = K × actual length of the column. In this problem (i.e: pined-pined column; K = 1 then, effective length = actual length = ***l***

7.3.2 Fixed-Fixed column

Figure (7-15) shows a fixed-fixed axially-loaded member (column) subject to the axial load (P) in its longitudinal direction x and buckled in y-direction with a maximum value of (δ) at column mid-length. The governing differential equation for this buckling can be derived from the equation of moment that induced by existence the force (P) and the buckling displacement (y). So that:

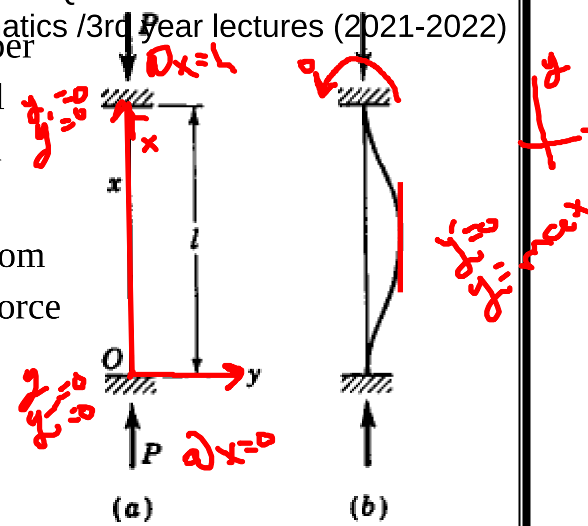


Figure (7-15) Fixed-fixed column

$M(x) = -Py + M_o$

Where, M_o = is the fixed end moment

$EIy'' = -Py + M_o \rightarrow y'' + \frac{P}{EI}y = \frac{M_o}{EI}$

$Let\ \frac{P}{EI} = k^2, then:$

$$y'' + k^2 y = \frac{M_o}{EI} \qquad (7-26)$$

yc — y-P

$y_t = y_c + y_p$

Equation (7-26) is classified as: Non-homogeneous 2nd Order D.E. It can be solved for the complementary solution by the method of undetermined coefficients using the characteristic equation as follows:

$m^2 + k^2 = 0 \rightarrow m^2 = -k^2 \rightarrow m_{1,2} = \mp ki\ (Where\ i = imajinary\ root = \sqrt{-1}$

Accordingly the complementary solution is:

$\boxed{y_c(x) = Asinkx + Bcoskx}$

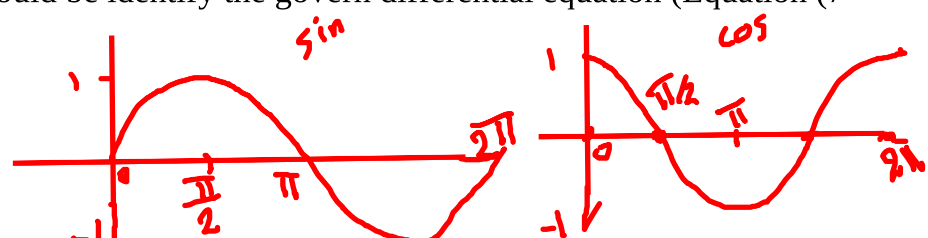
The particular solution for the non-homogeneous part of Equation (7-26) can be assumed as: $y_p(x) = C$ (constant) and should be identify the govern differential equation (Equation (7-26)), then:

$y'_p = 0$

$y''_p = 0$

$0 + k^2 C = \frac{M_o}{EI}$

$C\frac{P}{\cancel{EI}} = \frac{M_o}{\cancel{EI}} \rightarrow C = \frac{M_o}{P}$



Then the total solution is obtained by superposition of both complementary and particular solutions:

$y(x) = y_c(x) + y_p(x)$

$\boxed{y(x) = \boxed{A}sinkx + \boxed{B}cos\boxed{k}x + \frac{M_o}{P}}$

$y' = Ak\cos kx - Bk\sin kx$

To find the arbitrary constants A and B apply the following boundary conditions:

B.C

(a) $y'(0) = 0 \rightarrow 0 = Ak\cos(0) - Bk\sin(0) \rightarrow A = 0$

(b) $y(0) = 0 \rightarrow 0 = B\cos(0) + \frac{M_0}{P} \rightarrow B = -\frac{M_0}{P}$

$y(x) = -\frac{M_o}{P}coskx + \frac{M_o}{P}$

Or:

$$\boxed{y(x) = \frac{M_o}{P}(1 - coskx)}$$

To find the critical axial load at which buckling is occurred (*Euler's Load*) apply the third boundary condition, which is:

$y(l) = 0 = \frac{M_o}{P}(1 - coskl)$ This is can be achieved when

$kl = 2n\pi$

By squaring both sides:

$$k^2 l^2 = 4n^2\pi^2 \rightarrow \frac{P}{EI} l^2 = 4n^2\pi^2 \rightarrow \frac{P}{EI} = \frac{4n^2\pi^2}{l^2}$$

$$\boxed{\therefore P_{cr} = \frac{4\pi^2 EI}{l^2}}$$

Which is four times of the critical axial load of buckling for pined-pined column.

7.3.3 Fixed-pined column

Figure (7-16) shows a fixed-pined axially-loaded member (column) subject to the axial load (P) in its longitudinal direction x and buckled in y-direction with a maximum value of (δ) at column mid-length. The governing differential equation for this buckling can be derived from the equation of moment that induced by existence the force (P) and the buckling displacement (y) in addition to lateral force (F) at the pin support. So that:

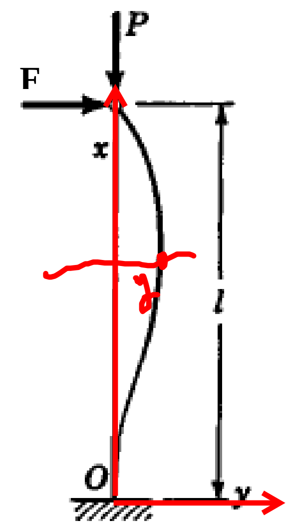


Figure (7-16) Fixed-fixed column

(+) $M(x) = -Py + Fx$

Where, F = is the lateral (or shear force at pin support)

$$EIy" = -Py + Fx \rightarrow y" + \boxed{\frac{P}{EI}} y = \frac{Fx}{EI}$$

$$Let \ \frac{P}{EI} = k^2, then:$$

y_c $\quad y" + k^2 y = \frac{Fx}{EI}$ $\quad y_p \qquad (7-27)$

Equation (7-27) is classified as: Non-homogeneous 2nd Order D.E. It can be solved for the complementary solution by the method of undetermined coefficients using the characteristic equation as follows:

$$m^2 + k^2 = 0 \rightarrow m^2 = -k^2 \rightarrow m_{1,2} = \mp ki \ (Where \ i = imajinary \ root = \sqrt{-1}$$

Accordingly the complementary solution is:

$$\boxed{y_c(x) = Asinkx + Bcoskx}$$

The particular solution for the non-homogeneous part of Equation (7-27) can be assumed as:

$y_p(x) = C + Dx$ and should be identify the govern differential equation (Equation (7-27)), then:

$y_p' = D$

$y_p'' = 0$

$$0 + k^2(C + Dx) = \frac{Fx}{EI}$$

$$\frac{P}{EI}(C + Dx) = \frac{Fx}{EI} \rightarrow C\frac{P}{EI} + D\frac{Px}{EI} = 0 + \frac{Fx}{EI}\ ,Mathematically\ this\ leads\ to:$$

$$C\frac{P}{EI} = 0 \rightarrow C = 0,\ and\ D\frac{Px}{EI} = \frac{Fx}{EI} \rightarrow D = \frac{F}{P}$$

$$\therefore y_p(x) = \frac{F}{P}x$$

Then the total solution is obtained by superposition of both complementary and particular solutions:

$$y(x) = y_c(x) + y_p(x)$$

$$\boxed{y(x) = Asinkx + Bcoskx + \frac{F}{P}x}$$

To find the arbitrary constants A and B apply the following boundary conditions:

(a) $y(0) = 0 \rightarrow B = 0$

(b) $y'(0) = 0, y'(x) = Acoskx + \frac{F}{P} \rightarrow y'(l) = 0 \rightarrow 0 = Ak\cos(kl) + \frac{F}{P} \rightarrow A = -\frac{F}{kPcoskl}$

$$\therefore y(x) = -\frac{F}{kPcoskl}sinkx + \frac{F}{P}x$$

Or:

$$y(x) = \frac{F}{P}\left(x - \frac{sinkx}{kcoskl}\right)$$

To find the critical axial load at which buckling is occurred (*Euler's Load*) apply the third boundary condition, which is:

$y(l) = 0 = \frac{F}{P}\left(l - \frac{tankl}{k}\right)$ This is can be achieved when:

$tankl = kl$

Which can be solved by trial and error to get:

$kl = 4.493\ rad$

By squaring both sides:

$$k^2l^2 = 20.19 \rightarrow k^2 = \frac{20.19}{l^2} \times \frac{\pi^2}{\pi^2} \rightarrow \frac{P}{EI} = \frac{\pi^2}{(0.7l)^2}$$

$$\boxed{\therefore P_{cr} = \frac{\pi^2 EI}{(0.7l)^2}}$$

Which is about two times of the critical axial load of buckling for pined-pined column.

7.3.4 Fixed-free column

Figure (7-17) shows a fixed-free axially-loaded member (column) subject to the axial load (P) in its longitudinal direction x and buckled in y-direction with a maximum value of (δ) at column mid-length. The governing differential equation for this buckling can be derived from the equation of moment that induced by existence the force (P) and the buckling displacement (y) in addition to lateral force (F) at the pin support. So that:

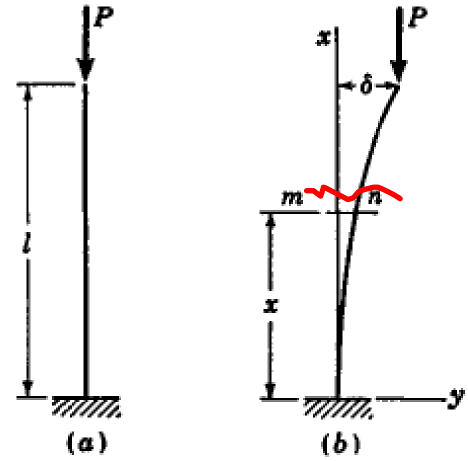


Figure (7-17) Fixed-free column

$$M(x) = -Py$$

$$EIy'' = -Py \rightarrow y'' + \frac{P}{EI}y = 0$$

$$Let \ \frac{P}{EI} = k^2, then:$$

$$y'' + k^2 y = 0 \qquad (7-28)$$

Equation (7-28) is classified as: Homogeneous 2nd Order D.E. It can be solved by the method of undetermined coefficients using the characteristic equation as follows:

$$m^2 + k^2 = 0 \rightarrow m^2 = -k^2 \rightarrow m_{1,2} = \mp ki \ (Where \ i = imajinary \ root = \sqrt{-1}$$

Accordingly the solution is:

$$\boxed{y(x) = Asinkx + Bcoskx}$$

To find the arbitrary constants A and B apply the following boundary conditions:

(a) $y(0) = 0 \rightarrow B = 0$

(b) $y'(l) = 0 \rightarrow , y'(x) = Acoskx \rightarrow 0 = Acoskl \rightarrow coskl = 0$ This is can be achieved when:

$$kl = \frac{n\pi}{2}$$

By squaring both sides:

$$k^2 l^2 = \frac{n^2\pi^2}{4} \rightarrow \frac{P}{EI} l^2 = \frac{n^2\pi^2}{4} \rightarrow \frac{P}{EI} = \frac{n^2\pi^2}{(2l)^2}$$

$$\boxed{P_{cr} = \frac{\pi^2 EI}{(2l)^2}}$$

Which is one quarter of the critical axial load of buckling for pined-pined column.

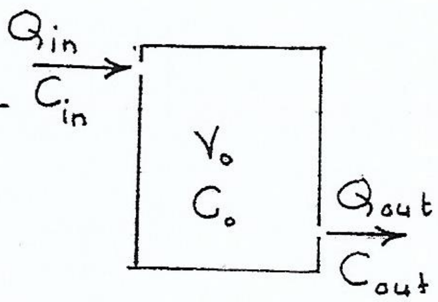
y'

Applications of First order Differential Equations

Application 1: Mixing of Liquids with Different Concentration

- For a tank contain a liquid with a specific concentration:

$$\text{Concentration} = \frac{\text{Amount of salts in solution}}{\text{Volume of solution}}$$



- The initial condition at time, $t=0$:

V_o = initial volume of tank (m^3, L, gal).

C_o = initial concentration of salts in tank (Kg/m^3, Kg/L, Ib/gal).

Therefore, the initial mass of salts in tank is

$$W_o = C_o V_o$$

- The change in mass of salts with time in the tank:

$$\frac{\text{change in mass}}{\text{with time}} = \frac{\text{Entered mass}}{\text{per unit time}} - \frac{\text{Exited mass}}{\text{per unit time}}$$

$$\therefore \frac{dw}{dt} = \sum Q_{in} C_{in} - \sum Q_{out} C_{out}$$

where:

W = mass of salt in solution at any time (t) units (Kg, Ib)

Q_{in} and Q_{out} = discharge of liquid interior and exterior of the tank. units (m^3/sec, L/sec, gal/min)

C_{in} and C_{out} = concentration of salts entered and exited of the tank. units (Kg/m^3, Kg/L, Ib/gal)

Note: If the liquid is pure, i.e ($C=0$).

To calculate C_{out}, there are two cases:

Case 1: If $Q_{in} \neq Q_{out}$ حالة تحصل عند غسل الخزان

$$C_{out} = \frac{w}{V_o + (Q_{in} - Q_{out})\, t}$$

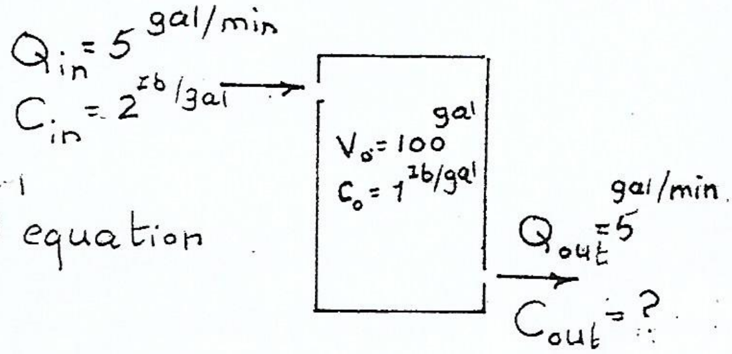
Case 2: If $Q_{in} = Q_{out}$ حالة عامة

$$C_{out} = \frac{w}{V_o}$$

Example 1: A tank is initially filled with (100 gal) of salt solution containing (1 Ib/gal). Fresh brine water containing (2 Ib/gal) run into the tank at a rate of (5 gal/min), and the mixture is assumed to be uniform then it runs out at the same rate. $Q_{in} = Q_{out}$

1. Find the amount of salt in the tank at any time, then
2. determine how long will it take for the amount of salt to be (150 Ib).

Solution:



ايجاد المعادلة التفاضلية

Step 1: Find the differential equation

$$\because Q_{in} = Q_{out}$$

$$\therefore C_{out} = \frac{W}{V_o} = \frac{W}{100}$$

$$\frac{dw}{dt} = \sum Q_{in} C_{in} - \sum Q_{out} C_{out}$$

$$\frac{dw}{dt} = (5)(2) - (5)\left(\frac{W}{100}\right)$$

$$\frac{dw}{dt} = \left(10 - \frac{W}{20}\right)$$

تحديد نوع المعادلة التفاضلية وحلها

3. Step 2: Specify the type of differential equation and solve.

$$\frac{dw}{10 - \frac{W}{20}} = dt$$

[The type of differential equation is variable separable]

To solve the differential equation

$$\int \frac{2x}{x^2}dx = \ln x^2$$

$$\int \frac{x}{x^2}dx = \frac{1}{2}\ln x^2$$

$$\int -\frac{1}{20} \frac{dw}{\left(10 - \frac{W}{20}\right)} = \int dt$$

B.C

ثابت التكامل

$$-20 \ln\left(10-\frac{w}{20}\right) = t + \boxed{C_1}$$

$$e^{3x+6} = e^{3x} \times e^{6}$$

$$\ln\left(10-\frac{w}{20}\right) = -\frac{t}{20} + \frac{C_1}{-20}$$

$$\ln\left(10-\frac{w}{20}\right) = \left(-\frac{t}{20} + C\right)$$

[$\frac{C_1}{-20}$ is constant and equal to c]

$$1 \quad e^{\ln\left(10-\frac{w}{20}\right)} = e^{\left(-\frac{t}{20}+C\right)}$$

$$10-\frac{w}{20} = e^{-\frac{t}{20}} * e^{c}$$

$$\boxed{10-\frac{w}{20} = A\, e^{-\frac{t}{20}}}$$

(where $A = e^{c}$).

B.C

تطبيق الشرط الابتدائي لايجاد الثابت

Step 3: Apply initial condition to find the constant

At $t=0 \Longrightarrow \boxed{W_o = C_o\, V_o}$

$$= (1)\,(100)$$

$$= 100^{Ib}$$

$$\therefore\ 10-\frac{100}{20} = A\, e^{0} \longrightarrow \text{where } e^{0} = 1$$

$$A = 5$$

$$\boxed{10-\frac{w}{20} = 5\, e^{-\frac{t}{20}}}$$

[The general equation for the amount of salts at any time]

ايجاد المطلوب

2 Step 4: Find the required

Find t for $w = 150^{Ib/gal}$

$$10-\frac{150}{20} = 5\, e^{-\frac{t}{20}}$$

$$2.5 = 5\, e^{-\frac{t}{20}}$$

$$\left[0.5 = e^{-\frac{t}{20}}\right] * \ln \qquad e^{\ln} = 1$$

$$\ln e^{\alpha t} \rightarrow \alpha t \ln e$$

$$\ln 0.5 = \ln e^{-\frac{t}{20}}$$

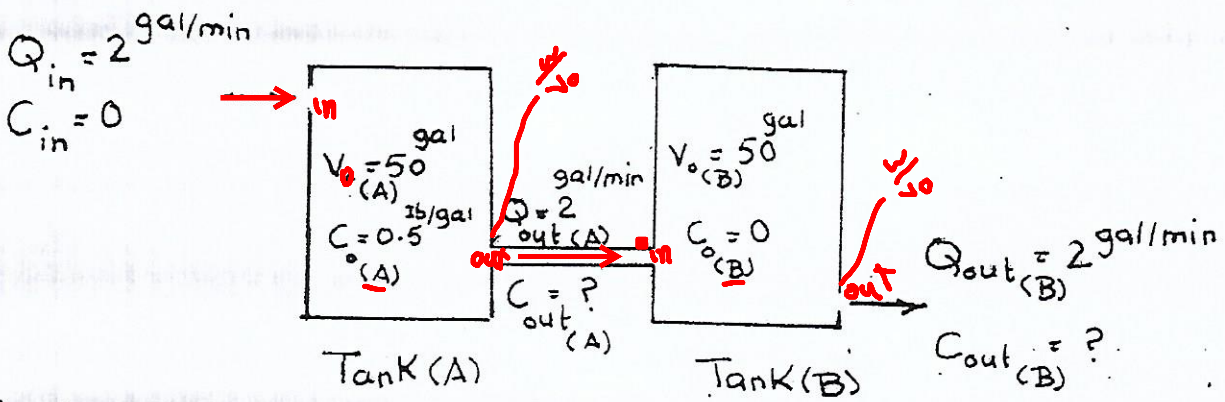
$$\ln 0.5 = -\frac{t}{20} \cancel{\ln e} \qquad (\ln e = 1)$$

$$t = -20 \ln 0.5$$

$$\therefore t = 13.863 \text{ min}$$

Example 2: Tank (A) initially contains (50^{gal}) V_0 of brine in which (0.5 Ib/gal) C_0 of salts are dissolved. A fresh pure water $C = 0\ in$ of ($2^{gal/min}$) Q_{in} enter to the tank (A) and the mixture (assumed to kept uniform) passes from tank (A) to tank (B) which is initially containing (50^{gal}) V_0 of fresh pure water C_0 at the same rate. The resulted mixture was also kept uniform, and run out from tank (B) at ($2^{gal/min}$) Q_{out}

Find the amount of salt W in tank (B) after one hour. $t = 60$ min.



Solution:

For Tank (A):

$$Q_{in} = Q_{out}$$

$$\underbrace{C_{out}}_{C_{in}\ (\text{tank } B)} = \frac{W_A}{V_0} = \frac{W_A}{50}$$

$$\frac{dW_A}{dt} = \sum Q_{in}\, C_{in} - Q_{out}\, C_{out}$$

$$\frac{dW_A}{dt} = \cancel{(2)(0)} - \cancel{(2)}^{1}\left(\frac{W_A}{\cancel{50}_{25}}\right) \Rightarrow \frac{dW_A}{dt} = \frac{-W_A}{25}$$

$$\frac{dW_A}{dt} = -\frac{W_A}{25}$$

$$\frac{dW_A}{W_A} = -\frac{dt}{25}$$

[First order variable seperable differential equation]

$$\int \frac{dW_A}{W_A} = -\int \frac{dt}{25}$$

$$\ln W_A = -\frac{t}{25} + C$$

$$e^{\ln W_A} = e^{-\frac{t}{25} + c}$$

1

$$W_A = e^{-\frac{t}{25}} * e^{c}$$

$$\therefore \boxed{W_A = K\, e^{-\frac{t}{25}}} \qquad (\text{where } K = e^{c})$$

At $t = 0 \implies W_o = C_o\, V_o = (0.5)(50) = 25 \text{ Ib}$

$$25 = K\, e^{0}$$

$$K = 25$$

$$\therefore \boxed{W_A = 25\, e^{-\frac{t}{25}}}$$

$$C_{out(A)} = \frac{W_A}{V_o} = \frac{25\, e^{-\frac{t}{25}}}{50} = \frac{1}{2}\, e^{-\frac{t}{25}}$$

For tank (B):

$$Q_{in} = Q_{out_{(A)}} = 2^{gal/min} \quad \text{and} \quad Q_{out} = 2^{gal/min}$$

$$C_{in} = C_{out_{(A)}} = \frac{1}{2} e^{-\frac{t}{25}} \quad \text{and} \quad C_{out} = \frac{W_B}{V_o} = \frac{W_B}{50}$$

$$\frac{dW_B}{dt} = \sum Q_{in} C_{in} - \sum Q_{out} C_{out}$$

$$\frac{dW_B}{dt} = (2)\left(\frac{1}{2} e^{-\frac{t}{25}}\right) - (\cancel{2})\left(\frac{W_B}{\cancel{50}\,25}\right)$$

$$\frac{dW_B}{dt} = e^{-\frac{t}{25}} - \frac{W_B}{25}$$

$$\frac{dW_B}{dt} + \frac{1}{25} W_B = e^{-\frac{t}{25}}$$

P, y, Q

[First order linear differential equation]

To solve the differential equation

$$P = \frac{1}{25} \quad \text{and} \quad Q = e^{-\frac{t}{25}}$$

حفظ المتغيرات [اثنين]

$$\boxed{\rho = e^{\int P\,dt}} = e^{\boxed{\int \frac{1}{25} dt}} = e^{\frac{t}{25}}$$

$$\boxed{\rho y = \int \rho Q \, dt + C}$$

[here y represent W_B]

$$\therefore \left(e^{\frac{t}{25}}\right)(W_B) = \int \left(e^{\frac{t}{25}}\right)\left(e^{-\frac{t}{25}}\right) dt + C$$

$$e^{\frac{t}{25}} W_B = \int dt + C$$

$$e^{\frac{t}{25}} W_B = t + C$$

60

$$\therefore \boxed{W_B = \frac{t + C}{e^{\frac{t}{25}}}}$$

To find the constant

At $t=0 \Longrightarrow W_{o_B} = C_o V_o = (0)(50) = 0$

$$0 = \frac{0+c}{e^0}$$

$\therefore c = 0$

$$\boxed{W_B = \frac{t}{e^{\frac{t}{25}}}}$$

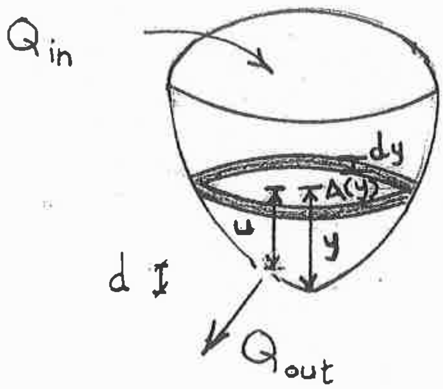
Now:

$$W_{B_{\text{at } t=60\,min}} = \frac{60}{e^{\frac{60}{25}}} = (60)\left(e^{-\frac{60}{25}}\right)$$

$$\therefore W_{B_{\text{at } t=60\,min}} = 5.443\ lb$$

Application 2: Discharge of the Tanks

- The purpose is to find the height of water at any time due to entry of water to the tank or exit from it due to gravity.



- The discharge exterior is

$$Q_{out} = C_d\, a \sqrt{2 g u}$$

C_d = [redacted] discharge coefficient

a = cross section of the hole. units (mm^2, m^2)

g = acceleration of gravity.

u = height of column of water above the hole, and

$$u = y - d$$

where d is the distance between the hole and bottom of the tank.

Note: If the hole at the bottom of the tank, i.e ($d = 0$).

- The change of volume of water in the tank at any time:

change of volume of water at any time = discharge interior − discharge exterior

$$\frac{dV}{dt} = \sum Q_{in} - \sum Q_{out}$$

but

$$dV = A(y)\, dy$$

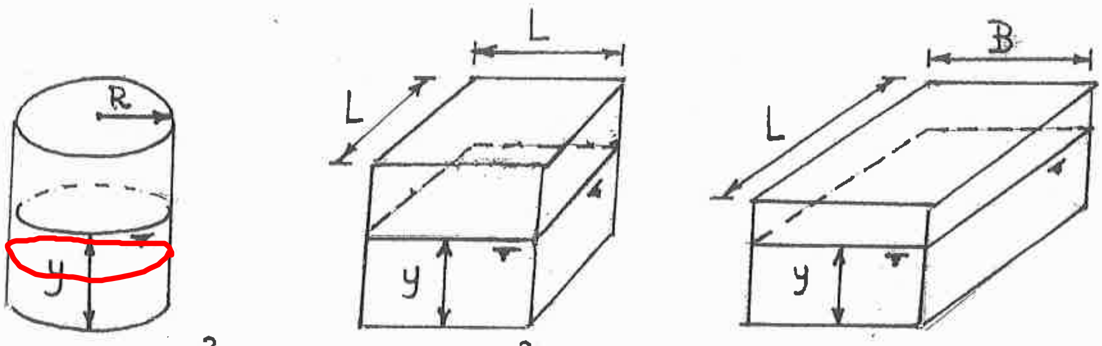
where: $A(y)$ is the horizontal cross section area at distance y from the bottom of the tank.

Therefore

$$A(y)\frac{dy}{dt} = \sum Q_{in} - \sum Q_{out}$$

حالات حساب مساحة المقطع الافقي $A(y)$

١. مساحة المقطع الافقي $A(y)$ ثابتة كما في متوازي المستطيلات أو مكعب أو اسطوانة قائمة.



$A(y) = \pi R^2$ $\qquad$ $A(y) = L^2$ $\qquad$ $A(y) = BL$

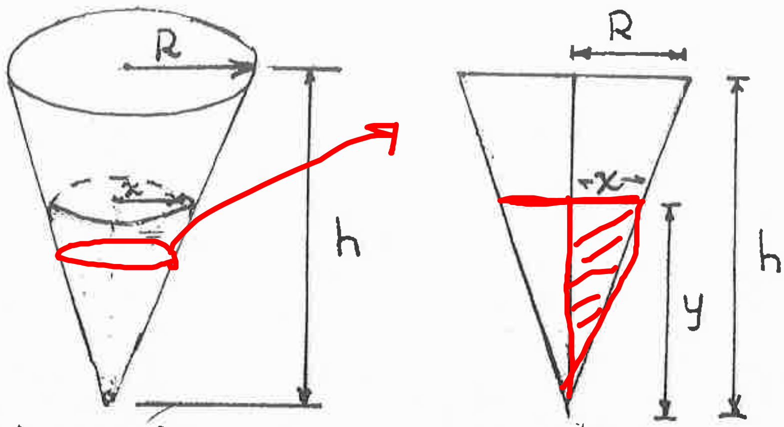
٢. مساحة المقطع الافقي $A(y)$ متغيرة مثل الاشكال المخروطية أو الكروية أو النصف كروية أو اسطوانة افقية وفي هذه الحالات يجب ايجاد علاقة بين ابعاد المقطع الافقي بدلالة (y) وذلك بالاستعانة بالمقطع العمودي ويتم ذلك بخطوتين:

أ. نرسم المقطع العمودي لايجاد علاقة بين ابعاد المقطع الافقي مع (y).

ب. ايجاد مساحة المقطع الافقي.

ايجاد المقطع العمودي حسب شكل الخزان:

(١) المقطع العمودي مثلث أو شبه منحرف (نسبة وتناسب).



الشكل المخروطي $\qquad$ مقطع عمودي

من المقطع العمودي:

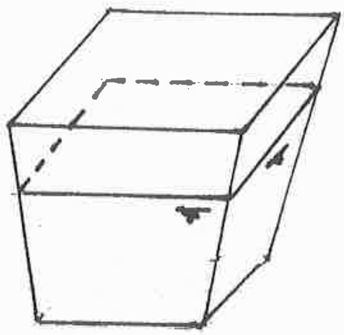
$$\frac{R}{h} = \frac{x}{y}$$

$$\therefore x = R\,\frac{y}{h}$$

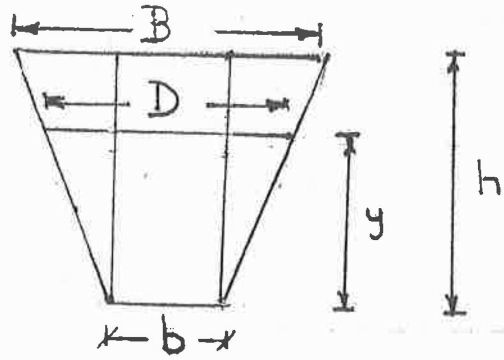
مساحة المقطع الافقي للشكل هو دائرة نصف قطرها (x) لذا :

$$A(y) = \pi x^2$$

$$= \pi \left(R\,\frac{y}{h}\right)^2$$



٣ب)



من المقطع العمودي:

$$\frac{B-b}{h} = \frac{D-b}{y}$$

$$\therefore D = \frac{y}{h}(B-b) + b$$

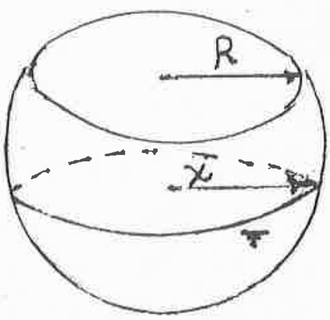
مساحة المقطع الافقي للشكل :

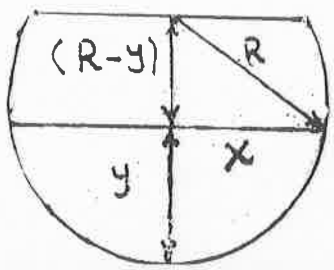
$$A(y) = D^2 \rightarrow \text{مستطيل}$$

$$= \left[\left(\frac{y}{h}\right)(B-b) + b\right]^2$$

(٢) المقطع العمودي دائرة أو نصف دائرة (نطبق قاعدة فيثاغورس)

٢ب)





من المقطع العمودي :

$$R^2 = x^2 + (R-y)^2$$

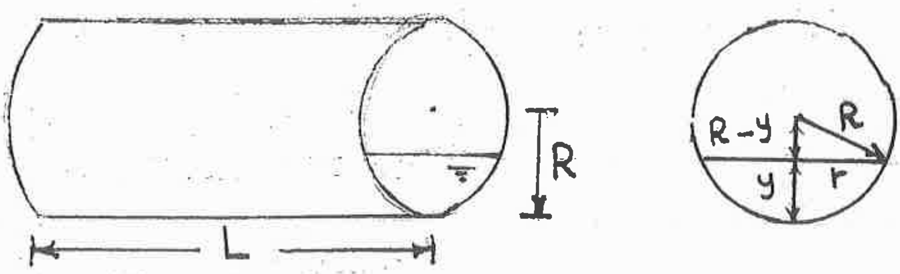
$$x^2 = R^2 - \left[R^2 - 2Ry + y^2\right]$$

$$\therefore \; x^2 = 2Ry - y^2$$

مساحة المقطع الافقي للشكل هو دائرة نصف قطرها (x) لذا :

$$A(y) = \pi x^2$$

$$= \pi \left(2Ry - y^2\right)$$

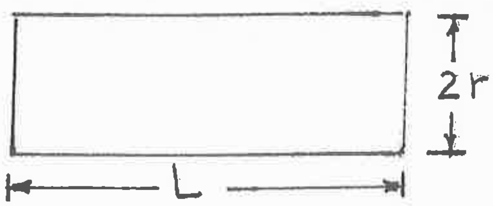


من المقطع العمودي :

$$R^2 = (R-y)^2 + r^2$$

$$\therefore \; r^2 = 2Ry - y^2$$

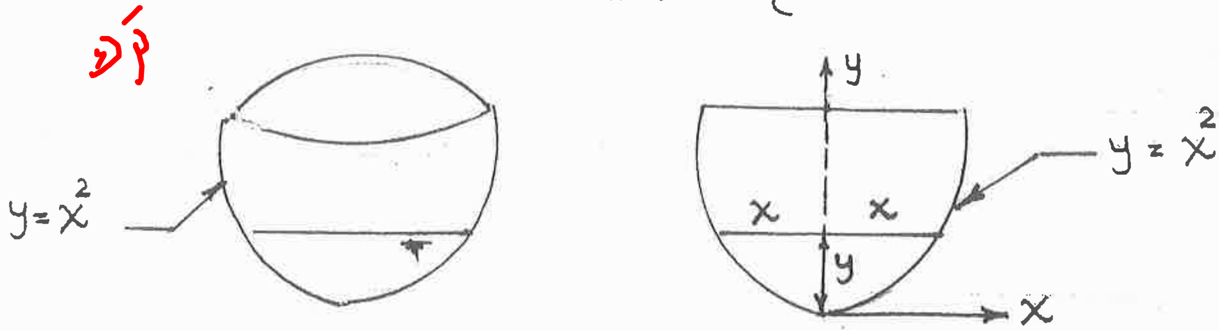
مساحة المقطع الافقي للشكل هو مستطيل :



$$A(y) = 2rL$$

$$= 2\sqrt{2Ry - y^2}\,(L)$$

(٣) المقطع العمودي عبارة عن دالة ، في هذه الحالة يجب ان نقطع معادلة الدالة للمقطع العمودي .



من المقطع العمودي :

$$y = x^2$$

$$\therefore\ x = \sqrt{y}$$

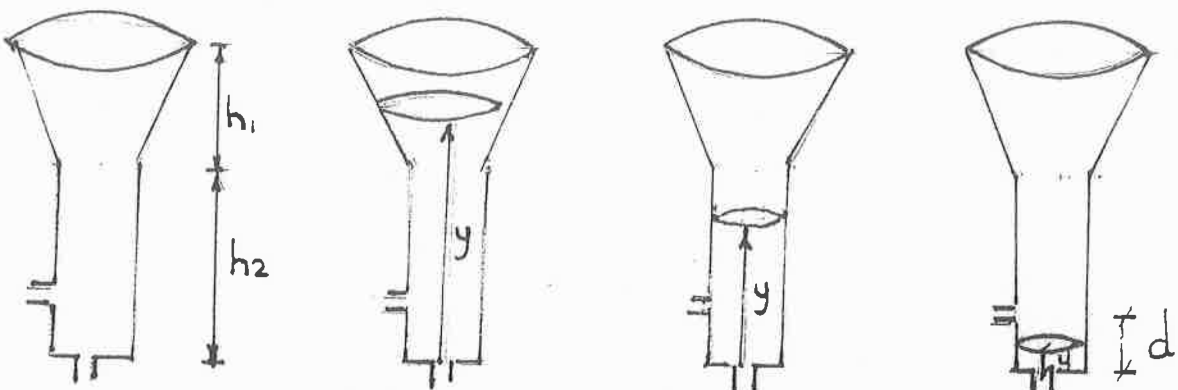
مساحة المقطع الافقي للشكل هو دائرة نصف قطرها (x) لذا :

$$A(y) = \pi x^2$$
$$= \pi y$$

ملاحظة :

١. في حالة الخزان الذي يحتوي على أكثر من مقطع عمودي فيقسم الحل لكل مقطع و يتم حساب $A(y)$ لكل منها ويطبق القانون لايجاد y .

٢. في حالة الخزان الذي يحتوي على عدة ثقوب في أماكن مختلفة فيقسم الحل لكل مجموعة ثقوب و نحسب $A(y)$ لكل حالة و يطبق القانون لايجاد y .



Example 1:

A cylindrical tank with radius of (2^m) and height of (4^m) is initially filled with water. At the bottom of the tank there is a hole of (2^{cm}) in diameter through which water drains under the influence of gravity. Find:

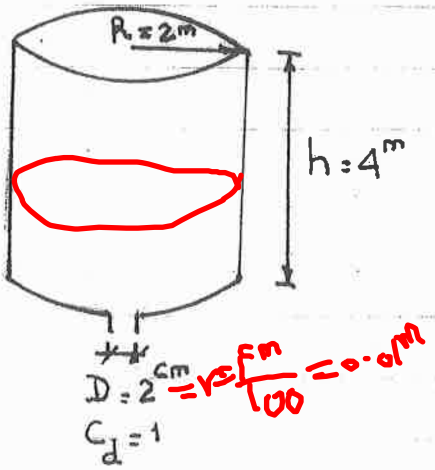
(1) The depth of water at any time. y & t

(2) The time $t=?$ for water to reach $y=1$ (1^m) in height.

(3) The time t required to empty $y=0$ the tank.

Solution:

(1) $A(y) = \pi R^2 = 4\pi$ (cross section of area is constant)



$D = 2^{cm} = r = \frac{1\,cm}{100} = 0.01^m$

$a = \pi r^2 = \pi (0.01)^2$ (area of the hole)

$$Q_{out} = C_d\, a \sqrt{2 g y}$$

$$= (1)\, \pi\, (0.01)^2 \sqrt{(2)(9.8)\, y}$$

$$A(y) \frac{dy}{dt} = \sum Q_{in} - \sum Q_{out}$$

$$\left[4\cancel{\pi} \frac{dy}{dt} = 0 - (1)\cancel{\pi}(0.01)^2 \sqrt{(2)(9.8)\, y} \right] \div \pi$$

$$4 \frac{dy}{dt} = -4.427 \times 10^{-4} \sqrt{y}$$

$$\frac{4}{4.427 \times 10^{-4}} \frac{dy}{\sqrt{y}} = -dt$$

$\frac{1}{\sqrt{y}} = \frac{1}{(y)^{1/2}} = (y)^{-1/2}$

$$\int 9035.46\, y^{-1/2} = -\int dt$$

$-\frac{1}{2} + \frac{2}{2} = \frac{1}{2}$

$$\cancel{9035.46} \frac{y^{1/2}}{\cancel{1/2}} = -t + c$$

$$\therefore\ 18070.92 \sqrt{y} = -t + c$$

Apply Boundary Condition :

I.C

At $t=0 \Longrightarrow y=4^{m}$

$$18070.92\sqrt{4} = -0 + C$$

$\therefore C = 36\,141.84$

The general equation will be:

$$\boxed{18070.92\sqrt{y} = -t + 36\,141.84}$$

(2) $y = 1^{m}$ $\quad t = ?$

solve

$$18070.92\sqrt{1} = -t + 36\,141.84$$

$\therefore t = 18070.92^{\,sec}$ $\quad (t = 5.02\ hr.)$

(3) To empty the tank, $y=0$

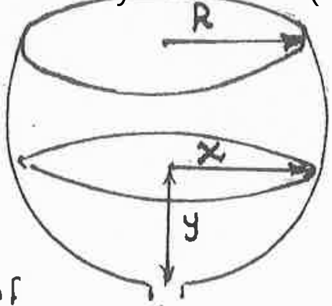
$$18070.92\sqrt{0} = -t + 36\,141.84$$

$\therefore t = \dfrac{36\,141.84^{\,sec}}{60\times 60}$ $\quad (t = 10.04\ hr.)$

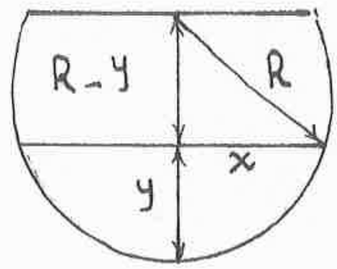
Example 2:

نصف كروي

A hemispherical tank of radius (R) is initially filled with water. At the bottom of the tank there is a hole of radius (r) through which the water drains under the influence of gravity. Find the depth of water at any time and determine the time required to empty the tank.



Radius of hole (r)



Vertical section

Solution:

From the vertical section:

$$R^2 = x^2 + (R - y)^2$$
$$x^2 = 2Ry - y^2$$

The horizontal cross section area is circle of radius (x):

$$A(y) = \pi x^2$$
$$= \pi (2Ry - y^2)$$

$$a = \pi r^2$$ (area of the hole)

$$Q_{out} = C_d\, a \sqrt{2gy}$$
$$= C_d\, \pi (r)^2 \sqrt{2gy}$$

$$A(y)\frac{dy}{dt} = \sum Q_{in} - \sum Q_{out}$$

تخلص للطرف الآخر

$$\left[\cancel{\pi}(2Ry - y^2)\frac{dy}{dt} = 0 - C_d \cancel{\pi}(r)^2 \sqrt{2g}\cancel{\sqrt{y}}\right] \times \frac{dt}{\pi\sqrt{y}}$$

$$\int (2Ry^{1/2} - y^{3/2})\,dy = \int -C_d\, r^2 \sqrt{2g}\; dt$$

$$\frac{2R\,y^{3/2}}{3/2} - \frac{y^{5/2}}{5/2} = -C_d\, r^2 \sqrt{29}\; t + c$$

Apply Boundary Condition:

At $t=0 \Longrightarrow y = R$

$$\frac{4}{3} R\, y^{3/2} - \frac{2}{5} y^{5/2} = -C_d\, r^2 \sqrt{29}\; t + c$$

$$\frac{4}{3} R^{5/2} - \frac{2}{5} R^{5/2} = -C_d\, r^2 \sqrt{29}\;(0) + c$$

$$\therefore c = \frac{14}{15} R^{5/2}$$

The general equation will be:

$$\frac{4}{3} R\, y^{3/2} - \frac{2}{5} y^{5/2} = -C_d\, r^2 \sqrt{29}\; t + \frac{14}{15} R^{5/2}$$

To empty the tank $\Longrightarrow y = 0$

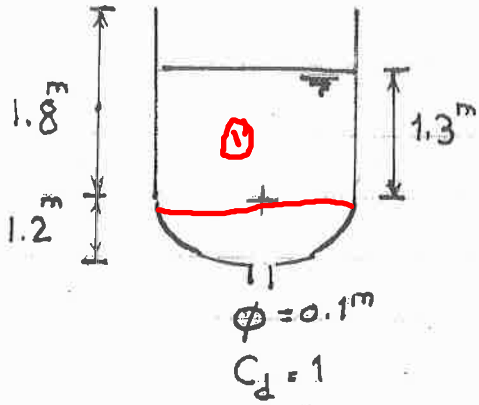
$$0 - 0 = -C_d\, r^2 \sqrt{29}\; t + \frac{14}{15} R^{5/2}$$

$$t = \frac{14\, R^{5/2}}{15\, C_d\, r^2 \sqrt{29}}$$

Example 3:

Find an equation to relate the depth of water in tank with respect to time.

Solution:



$$a = \pi r^2 = \pi (0.05)^2 = 0.0025\pi$$

$$Q_{out} = C_d\, a \sqrt{2gy}$$

$$= (1)(0.0025\pi)\sqrt{(2)(9.8)\,y}$$

$$= 0.0111\pi\sqrt{y}$$

Case 1: For $y > 1.2^m$

$$A(y) = \pi R^2 = \pi (1.2)^2 = 1.44\pi$$

$$A(y)\frac{dy}{dt} = \sum Q_{in} - \sum Q_{out}$$

$$\left[1.44\pi \frac{dy}{dt} = 0 - 0.0111\pi\sqrt{y}\right] \div 1.44\pi$$

$$\frac{dy}{dt} = -0.077\sqrt{y}$$

$$\int \frac{dy}{0.077\sqrt{y}} = -\int dt$$

$$129.87\,\frac{y^{1/2}}{1/2} = -t + c$$

$$\boxed{259.7\sqrt{y} = -t + c} \quad \text{--- Eq. (1)}$$

Apply Boundary Condition:

At $t=0 \Longrightarrow y=2.5^m$ نعوض في Eq. (1)

$$259.7\sqrt{2.5} = -0 + C$$

$$\therefore C = 410.685$$

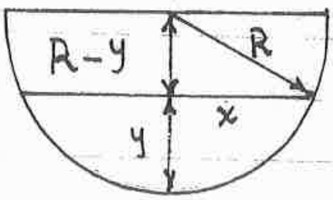
The general equation for case (1) will be:

$$\boxed{259.7\sqrt{y} = -t + 410.685}$$

Case 2: For $y < 1.2^m$

$$R^2 = x^2 + (R-y)^2$$

$$\therefore x^2 = 2Ry - y^2$$



$$A(y) = \pi x^2 = \pi(2Ry - y^2) = \pi(2.4y - y^2)$$

$$A(y)\frac{dy}{dt} = \sum Q_{in} - \sum Q_{out}$$

$$\left[\pi(2.4y - y^2)\frac{dy}{dt} = 0 - 0.0111\pi\sqrt{y}\right] \div \pi$$

$$(2.4y - y^2)\frac{dy}{dt} = -0.0111\sqrt{y}$$

$$\frac{2.4y - y^2}{0.0111\sqrt{y}}\,dy = -dt$$

$$\int \frac{2.4y}{0.0111\sqrt{y}}\,dy - \int \frac{y^2}{0.0111\sqrt{y}}\,dy = \int -dt$$

$$\int 216.2\, y^{1/2}\, dy - \int 90.1\, y^{3/2}\, dy = \int -dt$$

$$216.2\, \frac{y^{3/2}}{3/2} - 90.1\, \frac{y^{5/2}}{5/2} = -t + c$$

$$\boxed{144.1\, y^{3/2} - 36.04\, y^{5/2} = -t + c} \quad \text{--- Eq. (2)}$$

Apply Boundary Condition:

From the general equation of case (1), find:

$t = t_1$ at $y = 1.2^m$ (تمثل الحد $y = 1.2^m$ الفاصل)

$$259.7\sqrt{1.2} = -t_1 + 410.685$$

المعادلة (1) الأولى الماء يصل الى هذا الحد بعد 126.1 sec ومن بعدها تطبق معادلة (2)

$$\therefore t_1 = 126.1 \text{ sec}$$

Now; the boundary condition for case (2) is:

At $t = 126.1^{sec} \Longrightarrow y = 1.2^m$ نعوض في Eq. (2)

$$144.1\,(1.2)^{3/2} - 36.04\,(1.2)^{5/2} = -(126.1) + C$$

$$189.424 - 56.851 = -126.1 + C$$

$$\therefore C = 258.673$$

The general equation for case (2) will be:

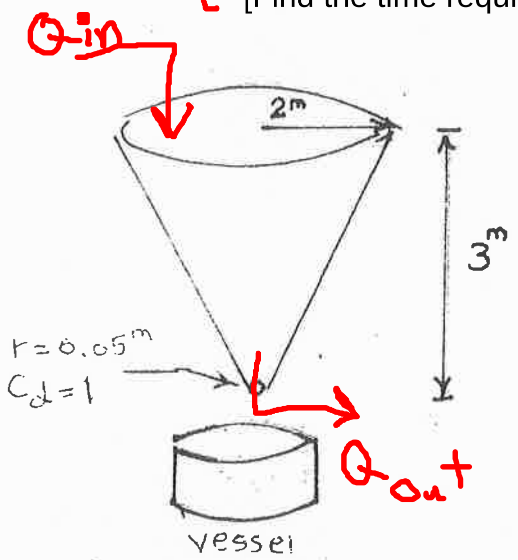
$$\boxed{144.1\sqrt{y^3} - 36.04\sqrt{y^5} = -t + 258.673}$$

I.C → t=0 → y=3

Example 4:

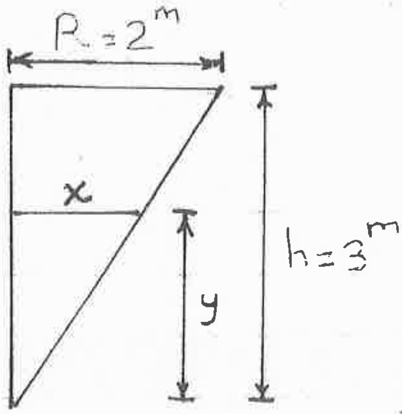
An inverted right circular cone tank with radius of (2^m) [R] and height of (3^m) is initially filled [y=h] with water. At the bottom of the tank there is a hole with a radius of (0.05^m) [a] through which the water drains under the influnce of gravity. If the hole is opened at ($t=0$) [t=0], find the time required [t] to fill a (~~1.5 m^3) vessel~~ as shown below:

[Find the time required if the water reach to 1.5m



Solution:

Step 1: Find the general solution for depth of water at any time.



$$\frac{2}{3} = \frac{x}{y}$$

$$\therefore\ x = \frac{2}{3}\, y$$

$$A(y) = \pi x^2$$

R

$$= \pi \left(\frac{2}{3}\, y\right)^2$$

h

$$= \frac{4}{9}\,\pi\, y^2 = 1.4\, y^2$$

$$a = \pi r^2 = \pi\,(0.05)^2$$

$$= 0.0025\,\pi$$

$Q_{in} = 0$

$Q_{out} = C_d \, a \sqrt{2 g y}$

$= (1)(0.0025\pi)\sqrt{(2)(9.8)y}$

$= 0.035\sqrt{y}$

$$A(y)\frac{dy}{dt} = \sum Q_{in} - \sum Q_{out}$$

$$\left[1.4\, y^2 \frac{dy}{dt} = 0 - 0.035\sqrt{y} \right] \times \frac{dt}{0.035\sqrt{y}}$$

$$40 \frac{y^2}{\sqrt{y}}\, dy = -dt$$

$y^2 \cdot y^{-\frac{1}{2}}$

$2 - \frac{1}{2} = \frac{4-1}{2} = \frac{3}{2}$

$$\int 40\, y^{3/2}\, dy = \int -dt$$

$\int x^2 = \frac{x^3}{3}$

$$40\left(\frac{2}{5}\, y^{5/2}\right) = -t + C_1$$

$\frac{3}{2} + \frac{2}{2} = \frac{5}{2}$

$\frac{y^{5/2}}{5/2} = \frac{2}{5} y^{5/2}$

$$\left[16\, y^{5/2} = -t + C_1 \right] \div 16$$

$$\boxed{\sqrt[2]{y^5} = -\frac{1}{16}t + C}$$

$\frac{C_1}{16}$

(where $C = \frac{C_1}{16}$)

Apply Boundary Condition:

At $t=0 \Longrightarrow y = 3^m$

$$\sqrt{3^5} = -\frac{1}{16}t + C$$

$\therefore C = 15.6$

The general equation will be:

$$\boxed{\sqrt{\underbrace{y}_{=1.5}{}^{5}} = -\frac{1}{16}\,t + 15.6} \Rightarrow$$

Step 2: Find the height of water so that the volume of water in the tank decreases by $1.5\ m^3$.

$$V_{vessel} = V_{total} - V_{at\ y=y_1}$$

$$1.5 = \left[\frac{1}{3}\pi (2)^2 (3)\right] - \left[\frac{1}{3}\pi \Big(\overbrace{\tfrac{2}{3}y_1}^{x}\Big)^2 y_1\right]$$

$$1.5 = 4\pi - \frac{4}{27}\pi y_1^3$$

$$12.56\, y_1^3 = (11.06)(27)$$

$$y_1^3 = \frac{298.62}{12.56}$$

$$\therefore\ y_1 = \sqrt[3]{23.775} = 2.873^{m}$$

Step 3: Find the time required to fill $1.5\ m^3$ of vessel, i.e at $y = 2.873^m$

$$\sqrt{(2.873)^5} = -\frac{1}{16}t + 15.6$$ (نعوض في المعادلة العامة)

$$14.027 = -0.0625\,t + 15.6$$

$$\therefore\ t = 25.168^{sec}$$

$$EI\frac{\pi^4}{4L^4}\sum_{n=1}^{\infty} n^4 y_n \cancel{\sin\frac{n\pi x}{L}} = \sum_{n=1}^{\infty}\frac{4P}{n\pi}\left(\sin\frac{n\pi}{2}\sin\frac{n\pi u}{L}\right)\cancel{\sin\frac{n\pi x}{L}}$$

$$y_n = \frac{PL^4}{n^5\pi^5 EI}\left(\sin\frac{n\pi}{2}\sin\frac{n\pi u}{L}\right) = \begin{cases} 0, & n=2,4,6 \\ \dfrac{PL^4}{n^5\pi^5 EI}\sin\dfrac{n\pi u}{L} & n=1,5,9 \\ -\dfrac{PL^4}{n^5\pi^5 EI}\sin\dfrac{n\pi u}{L} & n=3,7,11 \end{cases}$$

Since (u) is a small value the $\sin\frac{n\pi u}{L} \simeq \frac{n\pi u}{L}$ & this will makes:

$$y = \frac{PL^3u}{\pi^4 EI}\left(\sin\frac{\pi x}{L} - \frac{1}{81}\sin\frac{3\pi x}{L} + \frac{1}{625}\sin\frac{5\pi x}{L} + \cdots\right)$$

10- Partial differential Equations المعادلات التفاضلية الجزئية

Equations which contain one or more partial derivative are called Partial differential equation (P.D.E).
Therefore, there must be at least two independent variables & one dependent variable.
Let use take x, y to be 2 independent variable & z to be dependent variable.

Ex

$$\frac{\partial z}{\partial x} + z\frac{\partial z}{\partial y} = x^2$$

$f(y) = x \cdot z$

* $\boxed{\text{Order}}$: The order of partial differential equation is the highest partial derivative in the equation.
* $\boxed{\text{Degree}}$: The degree of the highest partial derivative in the equation is the degree of P.D.G.
* $\boxed{\text{linear}}$: If the dependent variable & their derivative from 1st degree
* $\boxed{\text{Homogenous}}$: If every term or part from the P.D.G $\boxed{\text{contain}}$ either dependent variable or their derivative, متجانسة

$$\lambda = \lambda \frac{\partial^2 y}{\partial x^2}$$

Examples

① $\frac{\partial z}{\partial x} + z\frac{\partial z}{\partial y} = x^2$ — 1st order, linear, 1st degree, Non homogenous

② $\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial u}{\partial z}\right)^3$ — 2nd order, linear, 1st degree, homogenous

③ $\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ — 2nd order, linear, 1st degree, homogenous

شروط Non-linear

$u\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 1$ — ١- المتغير التابع مضروب في احدى مشتقاته

$\sqrt{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}} = 0$ → ٢- المشتقة الجذورة او المرفوعة الى أس
المشتقة تحت الجذر

$(\ )^{1/2}$

$\left(\frac{\partial^2 u}{\partial x \partial y}\right)^3 \frac{\partial^2 u}{\partial y^2} = f(x)$ → المشتقة مرفوعة الى أس

٣- عندما يكون عندنا $(\sin u, e^{u}, \ln u, \cos u, \tan u, u^{n}, \sqrt{u})$

* The following equations are important P.D.E from 2nd order & homogenous :-

$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ ⟹ One dimensional wave Eq.

$\frac{du}{dt} = c^2 \frac{\partial^2 u}{\partial x^2}$ ⟹ one dimensional heat transfer Eq.

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ⟹ Two dimensional Laplaces Eq.

⊛ The suitable method to solve this problem it's (method of separation) or (Product method)

الطريقة المناسبة لحل هذه المسائل هي طريقة فصل المتغيرات وتسمى ايضا طريقة الضرب.

$u_{xx} = \frac{d^2u}{dx^2}$

u_y

D_x

u_x

$\frac{dy}{dx}$

ولايضاح هذه الطريقة يتم تفصيلها بالمثال التالي :-

Ex① Classify then Solve $u_x = x.y\, u_y$

$$\frac{\partial u}{\partial x} = x.y \frac{\partial u}{\partial y}$$

u = dep.

x, y = indep.

P.D.E

1st order

1st degree

Linear

① الآن نحل على إيجاد دالة لـ (x) نسميها (G) ودالة لـ (y) نسميها (F) وان حاصل ضرب الدالتين F, G يعطينا معادلة المتغير المعتمد (u) وبعبارة اخرى :-

$G(x) \longrightarrow$ (x) دالة لـ G

$F(y) \Longrightarrow$ (y) دالة لـ F

$F(y) = 3x + z$

$\frac{\partial F}{\partial x} = 3z\ 1$

$$\boxed{\text{Let } u(x,y) = G(x) \cdot F(y)}$$

* الآن $\frac{\partial u}{\partial x}$ مشتقة جزئية من المرتبة الأولى معناها تثبيت الدالة F واشتقاق الدالة G مرة واحدة اي

$$u_x = \frac{\partial u}{\partial x} = G'F \Longrightarrow \text{where } G' = \frac{dG}{dx}$$

$\frac{\partial u}{\partial y}$ معناها تثبيت الدالة G واشتقاق الدالة F مرة واحدة

$$u_y = \frac{\partial u}{\partial y} = G.F'$$

$u_{xy} = G'.F'$

$u_{tt} = G''$

$$\because u_x = x.y\, u_y$$

الآن نعوض الدوال الجديدة

$$[G'.F = x.y\, G.F']$$

نقسم المعادلة على G.F

$$\frac{G'}{G} = x.y \frac{F'}{F}$$

نقوم بعزل المتغيرات

$$\frac{1}{x}\frac{G'}{G} = y\frac{F'}{F} = \text{Constant} = \lambda$$

نقوم بمساواة المعادلة مع ثابت λ

$$\frac{1}{x}\frac{G'}{G} = \lambda$$

نأخذ الطرف الأيسر

$$\frac{1}{x}\frac{dG}{dx.G} = \lambda$$

$$\int \frac{dG}{G} = \lambda \int x\,dx$$

$(\ln G = \lambda \frac{x^2}{2} + C_1)$ نأخذ e للطرفين

$G = e^{\lambda \frac{x^2}{2} + C_1} \Longrightarrow G = e^{\lambda \frac{x^2}{2}} \cdot e^{C_1} \to A$

$$G = A\, e^{\lambda \frac{x^2}{2}}$$

$* \; y \cdot \frac{F'}{F} = \lambda \qquad \therefore F' = \frac{dF}{dy}$ الطرف الأيمن

$\therefore y \frac{dF}{dy \cdot F} = \lambda$

$\int \frac{dF}{F} = \lambda \int \frac{dy}{y} \Longrightarrow [\ln F = \lambda \ln y + C_2] * e$

$\Rightarrow F = e^{\lambda \ln y + C_2} \Longrightarrow F = e^{\lambda \ln y} \cdot e^{C_2} \to B$

$\Longrightarrow F = B e^{\lambda \ln y}$

* بعد الحصول على قيمة F , G نعوضها في العلاقة المكتوبة في الخطوة الأولى

$U(x,y) = G(x) \cdot F(y)$ $\quad$ $\boxed{\text{let } A \cdot B = k}$

$U = A e^{\lambda \frac{x^2}{2}} \cdot B e^{\lambda \ln y} \to e^{\lambda \ln y} = y^{\lambda}$

$$U = k \cdot e^{\lambda \frac{x^2}{2}} \cdot y^{\lambda}$$

Ex2 Solve $U_x + U_y = 2(x+y) \cdot U$

Sol: $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x+y) \cdot u$

$\left.\begin{array}{l} U : \text{dependant variable} \\ x, y : \text{independent variables} \end{array}\right\} \text{P.D.E}$

Let $U(x,y) = G(x) \cdot F(y)$

$\frac{\partial u}{\partial x} = G' \cdot F \quad \text{where } G' = \frac{\partial G}{\partial y}$

$U = G \cdot F$

$\Longrightarrow G' \cdot F + G \cdot F' = (2x+2y) \cdot G \cdot F] \div GF \Rightarrow \frac{G'}{G} + \frac{F'}{F} = 2x + 2y$

نقوم بعزل $(x, G, \acute{G})$ مع بعض
وال $(y, F, \acute{F})$ مع بعض

$$\frac{\acute{G}}{G} - 2x = 2y - \frac{\acute{F}}{F} = \text{constant } (\lambda)$$

$$\Rightarrow \frac{\acute{G}}{G} - 2x = \lambda$$

$$\frac{dG}{dx \cdot G} - 2x = \lambda$$ * نأخذ الطرف الأيسر

$$\frac{dG}{dx\,G} = \lambda + 2x$$

$$\int \frac{dG}{G} = \int \lambda\, dx + \int 2x\, dx$$

$$\ln G = \lambda x + x^2 + C_1] * e$$

$$G = e^{\lambda x + x^2 + C_1} \Rightarrow G = e^{C_1} \cdot e^{\lambda x + x^2} \rightarrow \text{let } A = e^{C_1}$$

$$\boxed{\therefore G = A e^{\lambda x + x^2}}$$

* نأخذ الطرف الأيمن

$$\Rightarrow 2y - \frac{\acute{F}}{F} = \lambda$$

$$2y - \lambda = \frac{\acute{F}}{F} \Rightarrow \frac{dF}{dy \cdot F} = 2y - \lambda$$

$$\Rightarrow \int \frac{dF}{F} = \int 2y \cdot dy - \int \lambda\, dy$$

$$\Rightarrow \ln F = y^2 - \lambda y + C_2] * e$$

$$\Rightarrow F = e^{y^2 - \lambda y + C_2}$$

$$F = e^{y^2 - \lambda y} \cdot e^{C_2} \rightarrow \text{let } B = e^{C_2}$$

$$F = B e^{y^2 - \lambda y}$$

$$\therefore u(x,y) = A e^{\lambda x + x^2} * B e^{y^2 - \lambda y} \rightarrow \text{let } A \cdot B = k$$

$$u(x,y) = k \cdot e^{\lambda(x-y) + x^2 + y^2}$$

Note ① For the same example given

$u(0,y) = 10\, e^{-4y+y^2}$

Find $u(0,2)$?

$\because u(x,y) = k \cdot e^{\lambda(x-y)+x^2+y^2}$

& given $u(0,y) = 10\, e^{-4y+y^2}$

$u = 10\, e^{-4y+y^2}$ $x=0$ $y=y$

$\therefore 10\, e^{-4y+y^2} = k \cdot e^{\lambda(0-y)+0+y^2}$

$10\, e^{-4y+y^2} = k \cdot e^{-\lambda y+y^2}$

$\therefore k = 10 \quad \lambda = 4$

$\therefore u(0,2) = 10\, e^{4(0-2)+0+(2)^2}$

$= 10\, e^{-4} = 10 * (2.718)^{-4}$

$= 0.1832$

Note ② Using the conditions

$u=2$ when $x=0$ & $y=0$

$u=15$ when $x=1$ & $y=0$

① $u=2$ when $x=0$ & $y=0 \Rightarrow 2 = k \cdot e^{\lambda(0-0)+0+0}$

$2 = k \cdot e^{0} \Rightarrow \therefore \boxed{k=2}$

② $u=15$ when $x=1$ & $y=0$

$15 = 2 \cdot e^{\lambda(1-0)+(1)^2+0} \Rightarrow \frac{15}{2} = e^{\lambda+1}$] * ln

$\Rightarrow \ln 7.5 = \lambda + 1 \Rightarrow 2.014 - 1 = \lambda$

$\therefore \lambda = 1.014$

وعليه اصبحت u بالشكل الاتي

$u(x,y) = 2 \cdot e^{1.014(x-y)+x^2+y^2}$

Ex2 $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$

Sol. z = dep. , y, x = independant.

let $z(x,y) = G(x).F(y)$.

$\frac{\partial z}{\partial x} = G'.F$ where $G' = \frac{\partial G}{\partial x}$

$\frac{\partial z}{\partial y} = G.F'$ where $F' = \frac{\partial F}{\partial y}$

$\frac{\partial^2 z}{\partial x^2} = G''.F$ where $G'' = \frac{\partial^2 G}{\partial x^2}$

$\Longrightarrow G''.F + 2G'.F - G.F' = 0] \div G.F$

$\frac{G''}{G} + 2\frac{G'}{G} - \frac{F'}{F} = 0$

$\frac{G''}{G} + 2\frac{G'}{G} = \frac{F'}{F} = \text{Constant}(\lambda)$

$\frac{G''}{G} + 2\frac{G'}{G} = \lambda \;] \times G$ * الطرف الايسر

$G'' + 2G' = \lambda G$

$G'' + 2G' - \lambda G = 0$

O.D.E, 2nd order 1st degree, linear, Homogenous w constant Coeff.

$m^2 + 2m - \lambda = 0$

$a = 1$, $b = 2$, $c = -\lambda$

$m_{1,2} = \frac{-2 \mp \sqrt{4 - 4 \times 1 \times -\lambda}}{2 \times 1}$

$m_1 = \frac{-2 + \sqrt{4 + 4\lambda}}{2}$

$m_2 = \frac{-2 - \sqrt{4 + 4\lambda}}{2}$

Case ① If $\lambda > 0$

* عند الوصول الى هذه المرحلة من الحل توجد فرضيات للحل والحل الصحيح يعتمد على نوع التطبيق

$$\text{let } \frac{\sqrt{4+4\lambda}}{2} = \mu$$

$$m_1 = -1 - \mu \quad , \quad m_2 = -1 - \mu$$

$$G(x) = C_1 e^{(-1+\mu)x} + C_2 e^{(-1-\mu)x}$$

Case 2 If $\lambda = 0$

$$m_1 = 0 \qquad m_2 = -2$$

$$G(x) = C_1 + C_2 e^{-2x}$$

Case ③ If $\lambda < 0$

$$m_{1,2} = -1 \mp \mu i$$

$$\alpha = -1 \qquad \beta = \mu$$

$$G(x) = e^{-x}(C_1 \cos \mu x + C_2 \sin \mu x)$$

$$\frac{F'}{F} = \lambda \Rightarrow \frac{dF}{dy \cdot F} = \lambda$$

* الطرف الأيمن

$$\int \frac{dF}{F} = \int \lambda \cdot dy \Rightarrow \ln F = \lambda y + C_3 \,]\times e$$

$$\Rightarrow F = e^{\lambda y + C_3} \Rightarrow F = e^{\lambda y} \cdot e^{C_3} \rightarrow = A$$

$$\therefore F = A \cdot e^{\lambda y}$$

$$\therefore Z = G \cdot F$$

$$\text{Case ①} \Rightarrow Z = [C_1 e^{(-1+\mu)x} + C_2 e^{(-1-\mu)x}] \cdot A e^{\lambda y}$$

$$\text{Case ②} \Rightarrow Z = [C_1 + C_2 e^{-2x}] A e^{\lambda y}$$

$$\text{Case ③} \Rightarrow Z = [e^{-x}(C_1 \cos \mu x + C_2 \sin \mu x)] \cdot A e^{\lambda y}]$$

System of Simultaneous Linear ordinary Differential Equations

معادلات تفاضلية آنية (لها أكثر من متغير تابع و متغير مستقل واحد).

For example, a system of simultaneous linear differential equation can be in the form:

$$2\frac{dx}{dt} + \frac{dy}{dt} - 4x - y = e^{t} \quad ---(1)$$

$$\frac{dx}{dt} + 3x + \frac{d^2y}{dt^2} + y = 0 \quad ---(2)$$

ملاحظة ، عدد المعادلات يكون مساوٍ لعدد التوابع .

خطوات الحل :

١. تحويل المشتقات الى المشغل التفاضلي (D) بحيث :

$$\frac{d}{dt} = D \quad , \quad \frac{d^2}{dt^2} = D^2$$

اذن تصبح المعادلات كالتالي :

$$2Dx + Dy - 4x - y = e^{t} \quad ---(1)$$

$$Dx + 3x + D^2y + y = 0 \quad ---(2)$$

٢. جمع الحدود المشتركة للمتغير التابع :

$$(2D - 4)x + (D - 1)y = e^{t} \quad ---(1)$$

$$(D + 3)x + (D^2 + 1)y = 0 \quad ---(2)$$

٣. نضع المعادلات بشكل مصفوفة   المعاملات

$$\begin{array}{cccc} \text{معاملات } x & \text{معاملات } y & \text{المتغيرات} & \text{الثوابت} \\ \end{array}$$

$$\begin{bmatrix} 2D-4 & D-1 \\ D+3 & D^2+1 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} e^t \\ 0 \end{Bmatrix}$$

٤. حل المصفوفة بأحدى الطرق مثل Crammer's Rule.

$D^2+3+\cancel{3D}-\cancel{D}$ $2D$

$$\begin{array}{cc} \text{معاملات } x & \text{معاملات } y \end{array}$$

$$\text{Det.} = \begin{vmatrix} 2D-4 & D-1 \\ D+3 & D^2+1 \end{vmatrix} = \left[(2D-4)(D^2+1)\right] - \left[(D+3)(D-1)\right]$$

$$= 2D^3 - 5D^2 - 1$$

$$\begin{array}{cc} \text{الثوابت} & \text{معاملات } y \end{array}$$

$$\text{Det.}x = \begin{vmatrix} e^t & D-1 \\ 0 & D^2+1 \end{vmatrix} = e^t(D^2+1) - 0$$

$$= D^2 e^t + e^t$$

$$= e^t + e^t = 2e^t$$

$$\begin{array}{cc} \text{معاملات } x & \text{الثوابت} \end{array}$$

$$\text{Det.}y = \begin{vmatrix} 2D-4 & e^t \\ D+3 & 0 \end{vmatrix} = 0 - (D+3)e^t$$

$$= -De^t - 3e^t$$

$$= -e^t - 3e^t = -4e^t$$

Now:

$$x = \frac{\text{Det.}x}{\text{Det.}} = \frac{2e^t}{2D^3 - 5D^2 - 1}$$

$$y = \frac{\text{Det.}y}{\text{Det.}} = \frac{-4e^t}{(2D^3 - 5D^2 - 1)}$$

$$(2D^3 - 5D^2 - 1)x = 2e^t \quad --- (3)$$

$$(2D^3 - 5D^2 - 1)y = -4e^t \quad --- (4)$$

٥. حل المعادلات اعلاه (3) و (4) لأيجاد :

$$x(t) = x_k + x_p$$

$$y(t) = y_k + y_p$$

Example 1:

Solve with a complete solution of $x(t)$ and $y(t)$ for the following system of differential equations

$$3x - \frac{dx}{dt} = 2\frac{dy}{dt} + e^t$$

$$\frac{dy}{dt} + 6y = \frac{dx}{dt}$$

مستقل t , y , x

Solution:

Step 1: ترتيب المعادلات بحيث يحتوي الطرف الايسر على المتغير التابع والطرف الايمن على المتغير المستقل أو الثابت .

$$3x - \frac{dx}{dt} - 2\frac{dy}{dt} = e^t \quad --- (1)$$

$$-\frac{dx}{dt} + \frac{dy}{dt} + 6y = 0 \quad --- (2)$$

تحويل المشتقات الى المشغل التفاضلي حيث :

$$\frac{d}{dt} = D$$

$$3x - Dx - 2Dy = e^{t} \qquad --- (1)$$

$$-Dx + Dy + 6y = 0 \qquad --- (2)$$

Step 2: جمع الحدود المشتركة للمتغير التابع

$$(3-D)x - 2Dy = e^{t} \qquad --- (1)$$

$$-Dx + (D+6)y = 0 \qquad --- (2)$$

Step 3: نضع المعادلات بشكل مصفوفة

$$\begin{array}{cc} x & y \\ \end{array}$$

$$\begin{bmatrix} 3-D & -2D \\ -D & D+6 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} e^{t} \\ 0 \end{Bmatrix}$$

Step 4: حل المصفوفة بأستخدام Crammer's Rule

$$\text{Det.} = \begin{vmatrix} 3-D & -2D \\ -D & D+6 \end{vmatrix} = \left[(3-D)(D+6)\right] - \left[(-D)(-2D)\right]$$

$$= -3D^{2} - 3D + 18$$

$$\text{Det. } x = \begin{vmatrix} e^{t} & -2D \\ 0 & D+6 \end{vmatrix} = e^{t}(D+6)$$

$$= De^{t} + 6e^{t}$$

$$= e^{t} + 6e^{t} = 7e^{t}$$

(تعني De^{t} المشتقة الاولى للدالة e^{t})

$$\text{Det. } y = \begin{vmatrix} 3-D & e^{t} \\ -D & 0 \end{vmatrix} = 0 - (-De^{t})$$

$$= De^{t}$$

$$= e^{t}$$

x & y in term of t

$$x = \frac{Det.\,x}{Det.} = \frac{7e^{t}}{-3D^2 - 3D + 18}$$

$$\therefore \; (-3D^2 - 3D + 18)\,x = 7e^{t} \qquad \text{--- (3)}$$

$$y = \frac{Det.\,y}{Det.} = \frac{e^{t}}{-3D^2 - 3D + 18}$$

$$\therefore \; (-3D^2 - 3D + 18)\,y = e^{t} \qquad \text{--- (4)}$$

Step 5: حل المعادلات اعلاه (3) و (4) لأيجاد

$$x(t) = x_c + x_p$$

and

$$y(t) = y_c + y_p$$

To find x_h and y_c:

From Eq. (3): $(-3D^2 - 3D + 18)\,x = 0 \qquad x \neq 0$

$$\therefore \; -3D^2 - 3D + 18 = 0$$
$$-3(D^2 + D - 6) = 0$$
$$D^2 + D - 6 = 0$$
$$(D - 2)(D + 3) = 0$$
$$\therefore \; D_1 = 2 \;,\; D_2 = -3$$

$$x_c = c_1 e^{2t} + c_2 e^{-3t}$$

And

$$\cancel{y_h = c_3 e^{2t} + c_4 e^{-3t}}$$

To find x_p:

Let $x_p = A e^{t}$

ملاحظة:

يقارن مع الحل المتجانس (x_h) بحيث لايوجد تكرار للدالـة المفروضة كحل خاص (x_p).

$$x'_p = A e^{t}$$

$$x''_p = A e^{t}$$

Sub. in Eq. (3):

$$-3Ae^{t} - 3Ae^{t} + 18Ae^{t} = 7e^{t}$$

$$12Ae^{t} = 7e^{t}$$

$$\therefore A = \frac{7}{12}$$

$$x_p = \frac{7}{12} e^{t}$$

$$\therefore x(t) = \underbrace{C_1 e^{2t} + C_2 e^{-3t}}_{y_c} + \underbrace{\frac{7}{12} e^{t}}_{y_p}$$

To find y_p:

Let $y_p = B e^{t}$

$$y'_p = B e^{t}$$

$$y''_p = B e^{t}$$

Sub. in Eq. (4):

$$-3Be^{t} - 3Be^{t} + 18Be^{t} = e^{t}$$

$$12Be^{t} = e^{t}$$

$$\therefore B = \frac{1}{12}$$

$$y_p = \frac{1}{12} e^{t}$$

$$\therefore y(t) = C_3 e^{2t} + C_4 e^{-3t} + \frac{1}{12} e^{t}$$

لايجاد علاقة بين الثوابت :

في حالة عدد المتغيرات التابعة اثنان نعوض في احدى المعادلات التفاضلية الاصلية ويتم التعويض في المعادلة التفاضلية الاسهل. اما في حالة عدد المتغيرات التابعة اكثر من اثنان فيتم زيادة عدد المعادلات التفاضلية التي يتم التعويض فيها.

Substitute in the original differential equation (2), so that:

$$\frac{dy}{dt} = 2C_3 e^{2t} - 3C_4 e^{-3t} + \frac{1}{12} e^{t}$$

$$\frac{dx}{dt} = 2C_1 e^{2t} - 3C_2 e^{-3t} + \frac{7}{12} e^{t}$$

$$\therefore \left(2C_3 e^{2t} - 3C_4 e^{-3t} + \cancel{\frac{1}{12}} e^{t}\right) + 6\left(C_3 e^{2t} + C_4 e^{-3t} + \cancel{\frac{1}{12}} e^{t}\right)$$

$$= 2C_1 e^{2t} - 3C_2 e^{-3t} + \cancel{\frac{7}{12}} e^{t}$$

Now:

$$2C_3 e^{2t} + 6C_3 e^{2t} = 2C_1 e^{2t}$$

$$8C_3 e^{t} = 2C_1 e^{2t}$$

$$\therefore C_3 = \frac{1}{4} C_1$$

$$-3C_4 e^{-3t} + 6C_4 e^{-3t} = -3C_2 e^{-3t}$$

$$3C_4 e^{-3t} = -3C_2 e^{-3t}$$

$$\therefore C_4 = -C_2$$

$$\therefore y(t) = \frac{1}{4} C_1 e^{2t} - C_2 e^{-3t} + \frac{1}{12} e^{t}$$

Example 2:

Determine a complete solution of the following system of differential equations

$$\frac{dw}{dx} + \frac{d^3z}{dx^3} + 2Z = 4w + x$$

$$\frac{dZ}{dx} + Z = 2w$$

Solution:

Step 1:

$$\frac{dw}{dx} - 4w + \frac{d^3z}{dx^3} + 2Z = x \quad \text{--- (1)}$$

$$-2w + \frac{dZ}{dx} + Z = 0 \quad \text{--- (2)}$$

Using differential operator:

$$\frac{d}{dx} = D \quad , \quad \frac{d^3}{dx^3} = D^3$$

$$\therefore \quad Dw - 4w + D^3Z + 2Z = x \quad \text{--- (1)}$$

$$-2w + DZ + Z = 0 \quad \text{--- (2)}$$

Step 2:

$$(D-4)w + (D^3+2)Z = x \quad \text{--- (1)}$$

$$-2w + (D+1)Z = 0 \quad \text{--- (2)}$$

Step 3:

$$\begin{bmatrix} D-4 & D^3+2 \\ -2 & D+1 \end{bmatrix} \begin{Bmatrix} w \\ Z \end{Bmatrix} = \begin{Bmatrix} x \\ 0 \end{Bmatrix}$$

Step 4:

$$\text{Det.} = \begin{vmatrix} D-4 & D^3+2 \\ -2 & D+1 \end{vmatrix} = \left[(D-4)(D+1)\right] - \left[(-2)(D^3+2)\right]$$

$$= 2D^3 + D^2 - 3D$$

$$\text{Det.}\,\omega = \begin{vmatrix} x & D^3+2 \\ 0 & D+1 \end{vmatrix} = x(D+1) - 0$$

$$= Dx + x$$

$$= 1 + x$$

$$\text{Det.}\,Z = \begin{vmatrix} D-4 & x \\ -2 & 0 \end{vmatrix} = 0 - (-2x)$$

$$= 2x$$

$$\omega = \frac{\text{Det.}\,\omega}{\text{Det.}} = \frac{1+x}{2D^3 + D^2 - 3D}$$

$$\therefore\ (2D^3 + D^2 - 3D)\,\omega = x + 1 \qquad \text{--- (3)}$$

$$Z = \frac{\text{Det.}\,Z}{\text{Det.}} = \frac{2x}{2D^3 + D^2 - 3D}$$

$$\therefore\ (2D^3 + D^2 - 3D)\,Z = 2x \qquad \text{--- (4)}$$

مهم جداً **Step 5:**

$$\omega(x) = \omega_h + \omega_p$$

$$Z(x) = Z_h + Z_p$$

To find ω_h and Z_h:

From Eq. (3): $(2D^3 + D^2 - 3D)\,\omega = 0 \qquad \omega \neq 0$

$$\therefore\ 2D^3 + D^2 - 3D = 0$$

$$D(2D^2 + D - 3) = 0$$
$$D\,(2D+3)(D-1) = 0$$
$$D_1 = 0 \quad D_2 = 1 \quad D_3 = \frac{-3}{2}$$

$$\therefore\ \omega_h = C_1 + C_2 e^{x} + C_3 e^{-\frac{3}{2}x}$$

And

$$Z_h = C_4 + C_5 e^{x} + C_6 e^{-\frac{3}{2}x}$$

<u>To find ω_p:</u>

Let $\omega_p = (Ax+B)x = Ax^2 + Bx$

$$\omega'_p = 2Ax + B$$
$$\omega''_p = 2A$$
$$\omega'''_p = 0$$

Sub. in Eq. (3):

$$2(0) + 2A - 3(2Ax+B) = x+1$$
$$2A - 6Ax - 3B = x+1$$

معاملات x $\quad -6Ax = x$

$$-6A = 1$$
$$\therefore\ A = \frac{-1}{6}$$

الثوابت $\quad 2A - 3B = 1$

$$2\left(-\frac{1}{6}\right) - 3B = 1$$

$$\therefore\ B = \frac{-4}{9}$$

$$\therefore \omega_p = -\frac{1}{6}x^2 - \frac{4}{9}x$$

$$\omega(x) = C_1 + C_2 e^{x} + C_3 e^{-\frac{3}{2}x} - \frac{1}{6}x^2 - \frac{4}{9}x$$

To find Z_p:

Let $Z_p = (Cx + E)x = Cx^2 + Ex$

$$Z'_p = 2Cx + E$$
$$Z''_p = 2C$$
$$Z'''_p = 0$$

Sub. in Eq.(4):

$$2(0) + 2C - 3(2Cx + E) = 2x$$
$$2C - 6Cx - 3E = 2x$$

معاملات x $\quad -6Cx = 2x$

$$-6C = 2$$
$$\therefore C = \frac{-1}{3}$$

الثوابت $\quad 2C - 3E = 0$

$$2\left(-\frac{1}{3}\right) - 3E = 0$$

$$\therefore E = \frac{-2}{9}$$

$$\therefore Z_p = \frac{-1}{3}x^2 - \frac{2}{9}x$$

$$Z(x) = C_4 + C_5 e^{x} + C_6 e^{-\frac{3}{2}x} - \frac{1}{3}x^2 - \frac{2}{9}x$$

To find a relationship between the constants, substitute in the original differential equation (2), so that:

$$\frac{dz}{dx} = 0 + C_5 e^{x} - \frac{3}{2} C_6 e^{-\frac{3}{2}x} - \frac{2}{3}x - \frac{2}{9}$$

$$\therefore -2\left(C_1 + C_2 e^{x} + C_3 e^{-\frac{3}{2}x} - \cancel{\frac{1}{6}x^2} - \cancel{\frac{4}{9}x}\right) + \left(C_5 e^{x} - \frac{3}{2} C_6 e^{-\frac{3}{2}x} - \cancel{\frac{2}{3}x} - \frac{2}{9}\right) + \left(C_4 + C_5 e^{x} + C_6 e^{-\frac{3}{2}x} - \cancel{\frac{1}{3}x^2} - \cancel{\frac{2}{9}x}\right) = 0$$

جمع الحدود المتشابهة:

$$(-2C_1 + C_4) + (-2C_2 + 2C_5)e^{x} + \left(-2C_3 - \frac{1}{2}C_6\right)e^{-\frac{3}{2}x} - \frac{2}{9} = 0$$

Now:

$$-2C_1 + C_4 - \frac{2}{9} = 0 \Longrightarrow C_4 = 2C_1 + \frac{2}{9}$$

$$(-2C_2 + 2C_5)e^{x} = 0 \qquad e^{x} \neq 0$$

$$\therefore -2C_2 + 2C_5 = 0 \Longrightarrow C_5 = C_2$$

$$\left(-2C_3 - \frac{1}{2}C_6\right)e^{-\frac{3}{2}x} = 0 \qquad e^{-\frac{3}{2}x} \neq 0$$

$$\therefore -2C_3 - \frac{1}{2}C_6 = 0 \Longrightarrow C_6 = -4C_3$$

$$\therefore Z(x) = 2C_1 + \frac{2}{9} + C_2 e^{x} - 4C_3 e^{-\frac{3}{2}x} - \frac{1}{3}x^2 - \frac{2}{9}x$$

$$= 2C_1 + C_2 e^{x} - 4C_3 e^{-\frac{3}{2}x} - \frac{1}{3}x^2 - \frac{2}{9}x + \frac{2}{9}$$

Example 3:

Find a solution of $x(t)$ for the following system of differential equations

$$\frac{dx}{dt} = x + y + t$$

$$\frac{dy}{dt} = 2x + y + z$$

$$\frac{dz}{dt} = 2y + z + e^{2t}$$

Solution:

Step 1:

$$\frac{dx}{dt} - x - y = t \quad \text{---(1)}$$

$$-2x + \frac{dy}{dt} - y - z = 0 \quad \text{---(2)}$$

$$-2y + \frac{dz}{dt} - z = e^{2t} \quad \text{---(3)}$$

Using differential operator:

$$\frac{d}{dt} = D$$

$\therefore$

$$Dx - x - y = t \quad \text{---(1)}$$

$$-2x + Dy - y - z = 0 \quad \text{---(2)}$$

$$-2y + Dz - z = e^{2t} \quad \text{---(3)}$$

Step 2:

$$(D-1)x - y = t \quad \text{---(1)}$$

$$-2x + (D-1)y - z = 0 \quad \text{---(2)}$$

$$-2y + (D-1)z = e^{2t} \quad \text{---(3)}$$

Step 3:

$$\begin{bmatrix} D-1 & -1 & 0 \\ -2 & D-1 & -1 \\ 0 & -2 & D-1 \end{bmatrix}\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} t \\ 0 \\ e^{2t} \end{Bmatrix}$$

Step 4:

$$\text{Det.} = \begin{array}{ccc} (+) & (-) & (+) \end{array}$$
$$\begin{vmatrix} D-1 & -1 & 0 \\ -2 & D-1 & -1 \\ 0 & -2 & D-1 \end{vmatrix}$$

$$= D-1\left\{\left[(D-1)(D-1)\right]-\left[(-2)(-1)\right]\right\} - (-1)\left\{\left[(-2)(D-1)\right] - 0\right\} + 0$$

$$= (D-1)\left[(D-1)^2 - 2\right] - 2(D-1)$$

$$= (D-1)\left[(D-1)^2 - 2 - 2\right]$$

$$= (D-1)\left(D^2 - 2D - 3\right)$$

$$= D^3 - 3D^2 - D + 3$$

$$\text{Det. } x = \begin{array}{ccc} (+) & (-) & (+) \end{array}$$
$$\begin{vmatrix} t & -1 & 0 \\ 0 & D-1 & -1 \\ e^{2t} & -2 & D-1 \end{vmatrix}$$

$$= t\left\{\left[(D-1)(D-1)\right]-\left[(-2)(-1)\right]\right\} - (-1)\left\{0 - \left[(e^{2t})(-1)\right]\right\} + 0$$

$$= t\,(D^2 - 2D - 1) + e^{2t}$$
$$= -2 - t + e^{2t}$$

$$x = \frac{\text{Det.}\,x}{\text{Det.}} = \frac{-2 - t + e^{2t}}{D^3 - 3D^2 - D + 3}$$

$$\therefore (D^3 - 3D^2 - D + 3)\,x = e^{2t} - t - 2 \qquad ---(4)$$

Step 5:

$$x(t) = x_h + x_p$$

To find x_h:

From Eq. (4): $(D^3 - 3D^2 - D + 3)\,x = 0 \qquad x \neq 0$

$$\therefore D^3 - 3D^2 - D + 3 = 0$$
$$(D-1)(D^2 - 2D - 3) = 0$$
$$(D-1)(D+1)(D-3) = 0$$
$$\therefore D_1 = 1 \quad D_2 = -1 \quad D_3 = 3$$

$$x_h = c_1 e^{t} + c_2 e^{-t} + c_3 e^{3t}$$

To find x_p:

Let $x_p = A e^{2t} + Bt + C$

$$\dot{x}_p = 2A e^{2t} + B$$
$$\ddot{x}_p = 4A e^{2t}$$
$$\dddot{x}_p = 8A e^{2t}$$

Sub. in Eq. (4):

$$8Ae^{2t} - 3(4Ae^{2t}) - (2Ae^{2t} + B) + 3(Ae^{2t} + Bt + C)$$

$$= e^{2t} - t - 2$$

$$\therefore 8Ae^{2t} - 12Ae^{2t} - 2Ae^{2t} + 3Ae^{2t} = e^{2t}$$

$$-3Ae^{2t} = e^{2t}$$

$$-3A = 1 \Longrightarrow A = \frac{-1}{3}$$

$$3Bt = -t$$

$$3B = -1 \Longrightarrow B = \frac{-1}{3}$$

$$-B + 3C = -2$$

$$-\left(-\frac{1}{3}\right) + 3C = -2$$

$$3C = \frac{-7}{3} \Longrightarrow C = \frac{-7}{9}$$

$$\therefore x_p = \frac{-1}{3}e^{2t} - \frac{1}{3}t - \frac{7}{9}$$

$$x(t) = c_1e^{t} + c_2e^{-t} + C_3e^{3t} - \frac{1}{3}e^{2t} - \frac{1}{3}t - \frac{7}{9}$$

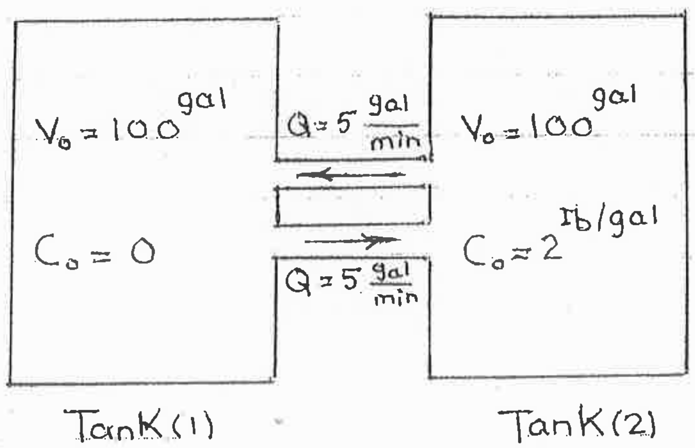
Applications of Simultanous Linear Differential Equations

Application 1: Salt Concentration in Tanks

Example 1:

Two tanks are connected as shown in figure, the first tank contains (100^{gal}) of pure water while the second tank contains (100^{gal}) of brine water with ($2^{Ib/gal}$) of salt dissolved in it.

The solution is circulated between the two tanks at a rate of ($5^{gal/min}$). If the mixture in both tanks is Kept uniforme by stirring, find the amount of salts in each tank with time.



Solution:

The change in mass of salts with time in tank:

$$\frac{dw}{dt} = \sum Q_{in}\, C_{in} - \sum Q_{out}\, C_{out}$$

Let x = the amount of salts in Tank (1)

y = the amount of salts in Tank (2)

Tank (1):

$$Q_{in\,Tank(1)} = Q_{out\,Tank(1)} = 5 \text{ gal/min}$$

$$C_{in\,Tank(1)} = \frac{W_{Tank(2)}}{V_{Tank(2)}} = \frac{y}{100}$$

$$C_{out\,Tank(1)} = \frac{W_{Tank(1)}}{V_{Tank(1)}} = \frac{x}{100}$$

$$\therefore \frac{dx}{dt} = 5\left(\frac{y}{100}\right) - 5\left(\frac{x}{100}\right)$$

$$\left[\frac{dx}{dt} = \frac{y}{20} - \frac{x}{20}\right] \times 20$$

$$20\frac{dx}{dt} = y - x \qquad \text{---} (1)$$

Tank (2):

$$Q_{in\,Tank(2)} = Q_{out\,Tank(2)} = 5 \text{ gal/min}$$

$$C_{in\,Tank(2)} = \frac{W_{Tank(1)}}{V_{Tank(1)}} = \frac{x}{100}$$

$$C_{out\,Tank(2)} = \frac{W_{Tank(2)}}{V_{Tank(2)}} = \frac{y}{100}$$

$$\therefore \frac{dy}{dt} = 5\left(\frac{x}{100}\right) - 5\left(\frac{y}{100}\right)$$

$$\left[\frac{dy}{dt} = \frac{x}{20} - \frac{y}{20}\right] \times 20$$

$$20\frac{dy}{dt} = x - y \qquad ---(2)$$

Step 1: ترتيب المعادلات (١) و (2) بحيث تحتوي الطرف الايسر على المتغير التابع والطرف الايمن على المتغير المستقل أو الثابت .

$$20\frac{dx}{dt} + x - y = 0 \qquad ---(1)$$

$$20\frac{dy}{dt} + y - x = 0 \qquad ---(2)$$

تحويل المشتقات الى المشغل التفاضلي بحيث :

$$\frac{d}{dt} = D$$

$$20Dx + x - y = 0 \qquad ---(1)$$

$$20Dy + y - x = 0 \qquad ---(2)$$

Step 2: جمع الحدود المشتركة للمتغير التابع

$$(20D+1)x - y = 0 \qquad ---(1)$$

$$-x + (20D+1)y = 0 \qquad ---(2)$$

step 3: نضع المعادلات بشكل مصفوفة

$$\begin{bmatrix} 20D+1 & -1 \\ -1 & 20D+1 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Step 4: حل المصفوفة بأستخدام Crammer's Rule

$$Det. = \begin{vmatrix} 20D+1 & -1 \\ -1 & 20D+1 \end{vmatrix} = (20D+1)^2 - 1$$
$$= \left[400D^2 + 40D + \cancel{1} - \cancel{1}\right] \div 40$$
$$= (10D^2 + D).40$$

$$Det.x = \begin{vmatrix} 0 & -1 \\ 0 & 20D+1 \end{vmatrix} = 0$$

$$x = \frac{Det.x}{Det.} = \frac{0}{40(10D^2+D)}$$

$$\therefore (10D^2 + D)x = 0 \qquad ---(3)$$

$$Det.y = \begin{vmatrix} 20D+1 & 0 \\ -1 & 0 \end{vmatrix} = 0$$

$$y = \frac{Det.y}{Det.} = \frac{0}{40(10D^2+D)}$$

$$\therefore (10D^2 + D)y = 0 \qquad ---(4)$$

Step 5: حل المعادلات (3) و (4) لايجاد $x(t)$ و $y(t)$

To find x_h and y_h:

From Eqn. (3): $(10D^2 + D)x = 0 \qquad x \neq 0$

$$\therefore 10D^2 + D = 0$$
$$D(10D+1) = 0$$
$$D_1 = 0 \quad D_2 = \frac{-1}{10} = -0.1$$

$$\therefore x(t) = C_1 e^{0} + C_2 e^{-0.1t}$$
$$= C_1 + C_2 e^{-0.1t}$$

From Eq. (4): $(10D^2 + D)\, y = 0$ $\qquad y \neq 0$

$$\therefore 10D^2 + D = 0$$
$$D(10D + 1) = 0$$
$$D_1 = 0 \qquad D_2 = \frac{-1}{10} = -0.1$$

$$\therefore y(t) = C_3 + C_4 e^{-0.1t}$$

To find a relationship between the constants, substitute in the original differential equation (1) in step 1 so that

$$\frac{dx}{dt} = -0.1\, C_2 e^{-0.1t}$$

$$\therefore 20(-0.1 C_2 e^{-0.1t}) + (C_1 + C_2 e^{-0.1t}) - (C_3 + C_4 e^{-0.1t}) = 0$$

نجمع الحدود المتشابهة

$$(-2C_2 + C_2 - C_4) e^{-0.1t} + (C_1 - C_3) = 0$$

Now:

$$(-2C_2 + C_2 - C_4) e^{-0.1t} = 0 \qquad e^{-0.1t} \neq 0$$
$$\therefore -2C_2 + C_2 - C_4 = 0$$
$$-C_2 - C_4 = 0 \Longrightarrow C_4 = -C_2$$

$$C_1 - C_3 = 0 \Longrightarrow C_3 = C_1$$

$$\therefore y(t) = C_1 - C_2 e^{-0.1t}$$

Step 6: Apply initial conditions

For Tank (1):

At $t=0 \implies x=0$

الخزان الاول في بداية الزمن
لا يحتوي على املاح

$$x(t) = C_1 + C_2 e^{-0.1t}$$
$$0 = C_1 + C_2$$
$$\therefore C_1 = -C_2$$

For Tank (2):

At $t=0 \implies C_o = \frac{y}{V_o}$

$$\therefore y = (2^{Ib/gal})(100^{gal})$$
$$= 200^{Ib}$$

$$y(t) = C_1 - C_2 e^{-0.1t}$$
$$200 = C_1 - C_2$$
$$200 = -C_2 - C_2$$
$$200 = -2C_2 \implies C_2 = \frac{-200}{2} = -100$$

$$\therefore C_1 = 100$$

Therefore:

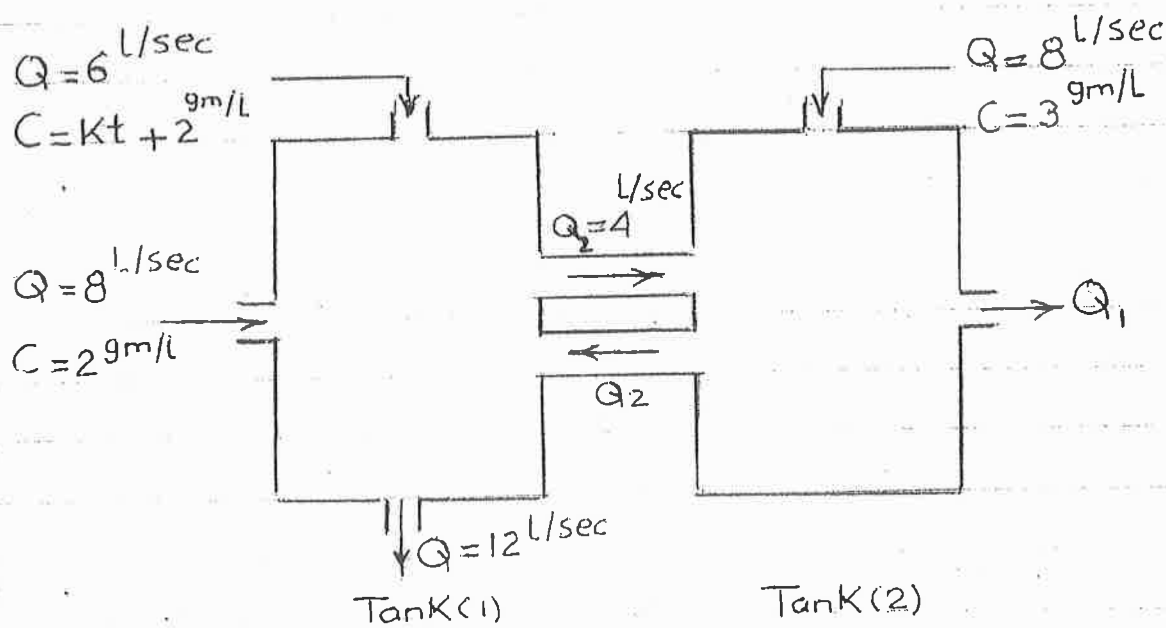
$$x(t) = 100 - 100 e^{-0.1t}$$
$$= 100\left(1 - e^{-0.1t}\right)$$

$$y(t) = 100 + 100 e^{-0.1t}$$
$$= 100\left(1 + e^{-0.1t}\right)$$

Example 2:

For the equilibrium system shown in figure, the liquid is circulated between the two tanks. If the volume of Tank (1) and Tank (2) are 250 Litters and 200 Litters respectively, find:

1. The discharge of liquid Q_1 and Q_2.
2. The differential equations of the system in matrix form which is required to determine the mass of salts at any time. where K is constant.



Solution:

1.
$$\sum Q_{in} = \sum Q_{out}$$

For all system

$$6 + 8 + 8 = 12 + Q_1 \Longrightarrow Q_1 = 10 \ \text{l/sec}$$

For Tank (1)

$$6 + 8 + Q_2 = 12 + 4 \Longrightarrow Q_2 = 2 \ \text{l/sec}$$

2. The change in mass of salts with time in tank:

$$\frac{dw}{dt} = \sum Q_{in}\, C_{in} - \sum Q_{out}\, C_{out}$$

Let x = the amount of salts in Tank (1)

y = the amount of salts in Tank (2)

<u>Tank (1):</u>

$$\frac{dx}{dt} = \left[6(Kt+2) + 8(2) + Q_2\left(\frac{y}{200}\right)\right] - \left[12\left(\frac{x}{250}\right) + 4\left(\frac{x}{250}\right)\right]$$

$$= 6Kt + 12 + 16 + 2\left(\frac{y}{200}\right) - \frac{12}{250}x - \frac{4}{250}x$$

$$\therefore \frac{dx}{dt} = 6Kt + 28 + \frac{1}{100}y - \frac{16}{250}x \qquad ---(1)$$

<u>Tank (2):</u>

$$\frac{dy}{dt} = \left[8(3) + 4\left(\frac{x}{250}\right)\right] - \left[Q_2\left(\frac{y}{200}\right) + Q_1\left(\frac{y}{200}\right)\right]$$

$$= 24 + \frac{4}{250}x - 2\left(\frac{y}{200}\right) - 10\left(\frac{y}{200}\right)$$

$$\therefore \frac{dy}{dt} = 24 + \frac{4}{250}x - \frac{12}{200}y \qquad ---(2)$$

<u>Step 1:</u> ترتيب المعادلات (1) و (2) بحيث يحتوي الطرف الايسر على المتغير التابع والطرف الايمن على المتغير المستقل او الثابت.

$$\frac{dx}{dt} + \frac{16}{250}x - \frac{1}{100}y = 6Kt + 28 \qquad ---(1)$$

$$\frac{dy}{dt} + \frac{12}{200}y - \frac{4}{250}x = 24 \qquad ---(2)$$

تحويل المشتقات الى المشغل التفاضلي بحيث :

$$\frac{d}{dt} = D$$

$$Dx + \frac{16}{250}x - \frac{1}{100}y = 6Kt + 28 \quad ---(1)$$

$$Dy + \frac{\cancel{12}\,6}{\cancel{200}\,100}y - \frac{4}{250}x = 24 \quad ---(2)$$

Step 2: جمع الحدود المشتركة للمتغير التابع

$$\left(D + \frac{16}{250}\right)x - \frac{1}{100}y = 6Kt + 28 \quad ---(1)$$

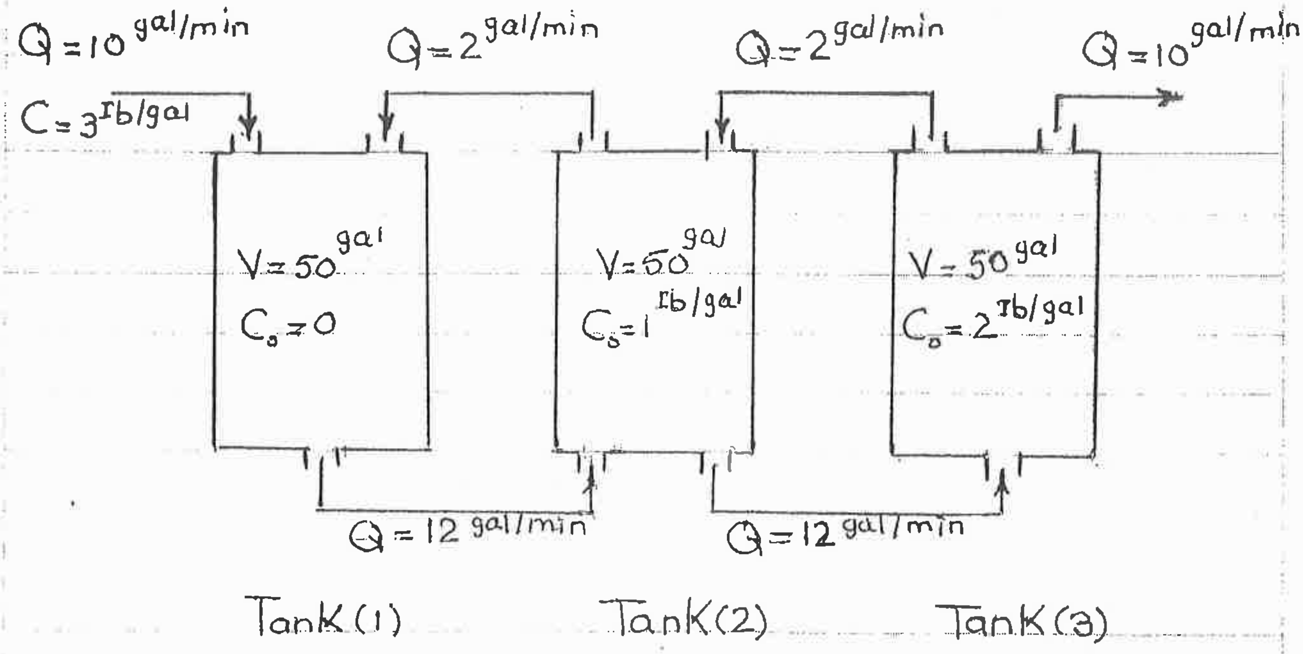
$$-\frac{4}{250}x + \left(D + \frac{6}{100}\right)y = 24 \quad ---(2)$$

Step 3: نضع المعادلات بشكل مصفوفة

$$\begin{bmatrix} D + \frac{16}{250} & -\frac{1}{100} \\ -\frac{4}{250} & D + \frac{6}{100} \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 6Kt + 28 \\ 24 \end{Bmatrix}$$

Example 3:

Find the differential equations of the system shown in figure in matrix form which is required to determine the mass of salts at any time. And write it's initial conditions.



Solution:

The change in mass of salts with time in tanks:

$$\frac{dw}{dt} = \sum Q_{in}\, C_{in} - \sum Q_{out}\, C_{out}$$

Let x = amount of salts in Tank (1)
y = amount of salts in Tank (2)
z = amount of salts in Tank (3)

Tank (1):

$$\frac{dx}{dt} = \left[10(3) + 2\left(\frac{y}{50}\right)\right] - 12\left(\frac{x}{50}\right)$$

$$\left[\frac{dx}{dt} = 30 + \frac{1}{25}y - \frac{6}{25}x\right] \times 25$$

$$25\frac{dx}{dt} = 750 + y - 6x \qquad \text{--- (1)}$$

Tank (2):

$$\frac{dy}{dt} = \left[12\left(\frac{x}{50}\right) + 2\left(\frac{z}{50}\right)\right] - (12+2)\left(\frac{y}{50}\right)$$

$$\left[\frac{dy}{dt} = \frac{6}{25}x + \frac{1}{25}z - \frac{7}{25}y\right] \qquad \times 25$$

$$25\frac{dy}{dt} = 6x + z - 7y \qquad ---(2)$$

Tank (3):

$$\frac{dz}{dt} = 12\left(\frac{y}{50}\right) - (10+2)\left(\frac{z}{50}\right)$$

$$\left[\frac{dz}{dt} = \frac{6}{25}y - \frac{6}{25}z\right] \qquad \times 25$$

$$25\frac{dz}{dt} = 6y - 6z \qquad ---(3)$$

Step 1:

$$25\frac{dx}{dt} + 6x - y = 750 \qquad ---(1)$$

$$25\frac{dy}{dt} + 7y - 6x - z = 0 \qquad ---(2)$$

$$25\frac{dz}{dt} + 6z - 6y = 0 \qquad ---(3)$$

Using differential operator: $\frac{d}{dt} = D$

$$25Dx + 6x - y = 750 \qquad ---(1)$$

$$25Dy + 7y - 6x - z = 0 \qquad ---(2)$$

$$25Dz + 6z - 6y = 0 \qquad ---(3)$$

Step 2:

$$(25D+6)x - y = 750 \qquad ---(1)$$
$$-6x + (25D+7)y - z = 0 \qquad ---(2)$$
$$-6y + (25D+6)z = 0 \qquad ---(3)$$

Step3:

$$\begin{bmatrix} 25D+6 & -1 & 0 \\ -6 & 25D+7 & -1 \\ 0 & -6 & 25D+6 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 750 \\ 0 \\ 0 \end{Bmatrix}$$

Now:

The initial conditions are:

For Tank (1):

At $t=0 \implies C_o = \dfrac{x}{V}$ where $C_o = 0$

$\therefore \quad x = 0$

For Tank (2):

At $t=0 \implies C_o = \dfrac{y}{V}$

$\therefore \quad y = (1^{\,Ib/gal})(50^{\,gal}) = 50^{\,Ib}$

For Tank (3):

At $t=0 \implies C_o = \dfrac{z}{V}$

$\therefore \quad z = (2^{\,Ib/gal})(50^{\,gal}) = 100^{\,Ib}$

$$\therefore x(12) = 137.3 + e^{-0.075(12)}\left[-137.3\cos\frac{\sqrt{3}}{40}(12) - 6.86\left(\frac{\sqrt{3}}{40}(12)\right)\right]$$
$$= 87\ Ib$$

$$y(12) = 111.78\ Ib$$

$$z(12) = 157.91\ Ib$$

8 - Foruier Series

نعني بال (F.S) هي امكانية تحويل دالة بدلالة الزمن (F(t)) الا مجموعة دوال بدلالة ال (Sin) و (Cos)

* periodic Function :- if the value of the function repeats itself at regular intervals of t we call that function is periodic.

$$f(t) = f(t+2p)$$

where: $2p$ = period

* The Fourier series is :

$$f(t) = \frac{a_0}{2} + a_1\cos\frac{\pi t}{p} - a_2\cos\frac{2\pi t}{p} + \dots + a_n\cos\frac{n\pi t}{p} + \dots$$
$$+ b_1\sin\frac{\pi t}{p} + b_2\sin\frac{2\pi t}{p} + \dots + b_n\sin\frac{n\pi t}{p} + \dots$$

where :-

a_0, a_n, b_n = Euler Fourier Coefficients

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi t}{p}\right) + b_n\sin\left(\frac{n\pi t}{p}\right)\right) \quad \text{--- حفظ}$$

$$a_0 = \frac{1}{p}\int_{b_1}^{b_2} f(t)\,dt$$

$$a_n = \frac{1}{p}\int_{b_1}^{b_2} f(t)\cdot\cos\left(\frac{n\pi t}{p}\right)dt$$

$$b_n = \frac{1}{p}\int_{b_1}^{b_2} f(t)\sin\left(\frac{n\pi t}{p}\right)dt.$$

Matches are very important

1- $\sin\alpha \sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$

2- $\sin\alpha \cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

3- $\cos\alpha \sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

4- $\cos\alpha \cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$

5- $\sin^2\alpha = \frac{1}{2}(1-\cos2\alpha)$

6- $\cos^2\alpha = \frac{1}{2}(1+\cos2\alpha)$

7- $\sin2\alpha = 2\sin\alpha\cos\alpha$

8- $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

9- $\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

10- $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

11- $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

12- $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

13- $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

14- $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

15- $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

* $\sin 0 = 0$
* $\cos 0 = 1$
* $\sin n\pi = 0$
* $\cos n\pi \xrightarrow{(-1)^n}$ $\rightarrow -1$ if n = odd فردي ; $\rightarrow 1$ if n = even زوجي

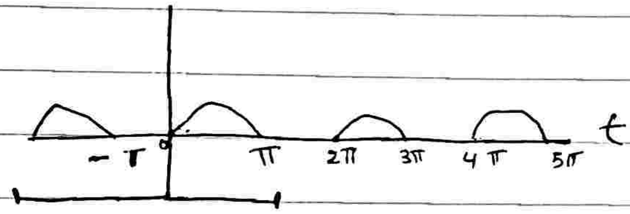
* $(\cos n\pi - 1) \Rightarrow$ $\rightarrow -2$ if n = odd ; $\rightarrow 0$ if n = even

$\cancel{a_0}, \cancel{a_n}, \cancel{b_n}$

Ex 1 what is the Fourier Expansion of the periodic function where definition in one period is

$$f(t) = \begin{cases} 0 & \text{for } -\pi < t < 0 \\ \sin t & \text{for } 0 < t < \pi \end{cases}$$

Sol.



$$2P = 2\pi \Rightarrow P = \pi$$

$$a_0 = \frac{1}{P}\int_{b_1}^{b_2} f(t)\,dt = \frac{1}{\pi}\left[\cancel{\int_{-\pi}^{0} 0\,dt} + \int_0^{\pi} \sin t\,dt\right]$$

$$= \frac{1}{\pi}\left[-\cos t\right]_0^{\pi}$$

$$= \frac{-1}{\pi}\left[\cos\pi - \cos 0\right]$$

$$= -\frac{1}{\pi}\left[-1-1\right] = \frac{2}{\pi}$$

$$a_n = \frac{1}{P}\int_{b_1}^{b_2} f(t)\cos\frac{n\pi t}{P}\,dt$$

$$= \frac{1}{\pi}\left[\cancel{\int_{-\pi}^{0} 0\cos\frac{n\pi t}{P}\,dt} + \int_0^{\pi}\sin t \cdot \cos\frac{n\pi t}{P}\,dt\right]$$

$$= \frac{1}{\pi}\int_0^{\pi}\sin t\cos nt\,dt$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi}\sin(1+n)t + \sin(1-n)t\right]dt$$

$$a_n = \frac{1}{2\pi}\left[\frac{-\cos(1+n)t}{1+n} - \frac{\cos(1-n)t}{1-n}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(1+n)t}{1+n} - \frac{\cos(1-n)t}{1-n}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{-(1-n)\cos(1+n)t - (1+n)\cos(1-n)t}{(1+n)(1-n)}\right]_0^{\pi}$$

$$= \frac{1}{2\pi(1-n^2)}\left[-\cos(1+n)t + n\cos(1+n)t - \cos(1-n)t - n\cos(1-n)t\right]_0^{\pi}$$

$$= \frac{1}{2\pi(1-n^2)}\Big[\big(-\cos(1+n)\pi + n\cos(1+n)\pi - \cos(1-n)\pi - n\cos(1-n)\pi\big) - \big(-\cos(1+n)0 + n\cos(1+n)0 - \cos(1-n)0 - n\cos(1-n)0\big)\Big]$$

$$= \frac{1}{2\pi(1-n^2)}\Big[\big(-\cos(\pi+n\pi) + n\cos(\pi+n\pi) - \cos(\pi-n\pi) - n\cos(\pi-n\pi)\big) - (-1+n-1-n)\Big]$$

$$= \frac{1}{2\pi(1-n^2)}\Big[-(\cos\pi\cos n\pi - \sin\pi\sin n\pi) + n(\cos\pi\cos n\pi - \sin\pi\sin n\pi) - (\cos\pi\cos n\pi + \sin\pi\sin n\pi) - n(\cos\pi\cos n\pi + \sin\pi\sin n\pi) + 2\Big]$$

$$= \frac{1}{2\pi(1-n^2)}(\cos n\pi - n\cos n\pi + \cos n\pi + n\cos n\pi + 2)$$

$$= \frac{1}{2\pi(1-n^2)}(2\cos n\pi + 2) = \frac{\cos n\pi + 1}{\pi(1-n^2)}$$

n	a_n
1	$a_1 = \frac{\cos\pi + 1}{\pi(1-1)} = 0$
2	$a_2 = \frac{2}{-3\pi}$
3	$a_3 = 0$
⋮	

To find a_1 :-

① $\lim_{n\to 1} \frac{\cos n\pi + 1}{\pi(1-n^2)}$ Use lopetal's rule

$$a_1 = \frac{-n\sin n\pi + 0}{\pi(1-2n)} = \frac{-1\,n\sin\pi}{\pi(1-2(1))} = \frac{0}{-\pi} = 0$$

② $a_1 = \frac{1}{\pi}\int_0^{\pi} \sin t\cos t\,dt = \frac{1}{2\pi}\left[(\sin t)^2\right]_0^{\pi}$

$$= \frac{1}{2\pi}\left[(\sin\pi)^2 - (\sin 0)^2\right] = \frac{0}{2\pi} = 0$$

$$b_n = \frac{1}{P}\int_{b_1}^{b_2} f(t)\sin\frac{n\pi t}{P}\,dt$$

$$= \frac{1}{\pi}\Big[\int_{-\pi}^{0} 0\,\sin nt\,dt + \int_{0}^{\pi}\sin t\,\sin nt\,dt\Big]$$

$$= \frac{1}{\pi}\int_{0}^{\pi}\sin t\,\sin nt\,dt = \frac{1}{\pi}\int_{0}^{\pi}\Big[\frac{1}{2}\big(\cos(1-n)t - \cos(1+n)t\big)\Big]dt$$

$$= \frac{1}{2\pi}\Big[\frac{1}{(1-n)}\sin(1-n)t - \frac{1}{(1+n)}\sin(1+n)t\Big]_0^{\pi}$$

$$= \frac{1}{2\pi}\Big[\frac{1}{1-n}\cancel{\sin(1-n)\pi}^{\,0} - \frac{1}{(1+n)}\cancel{\sin(1+n)\pi}^{\,0}\Big] = 0$$

$\therefore b_n = 0 \Rightarrow$ لجميع قيم n عدا (1) $n=1$

$$b_1 = \frac{1}{\pi}\int_{0}^{\pi}\sin t\,\sin\frac{n\pi t}{\pi} = \frac{1}{\pi}\int_{0}^{\pi}\sin^2 t\,dt$$

$$= \frac{1}{\pi}\int_{0}^{\pi}\frac{1}{2}(1-\cos 2t)\,dt = \frac{1}{2\pi}\Big(t - \frac{1}{2}\sin 2t\Big)_0^{\pi}$$

$$= \frac{1}{2\pi}\Big[\Big(\pi - \frac{1}{2}\cancel{\sin 2\pi}^{\,0}\Big) - \Big(0 - \frac{1}{2}\cancel{\sin(0)}^{\,0}\Big)\Big] = \frac{\pi}{2\pi} = \frac{1}{2}$$

$$f(t) = \frac{a_0}{2} + \sum b_n \sin\frac{n\pi}{P}t + \sum a_n\cos\frac{n\pi t}{P}$$

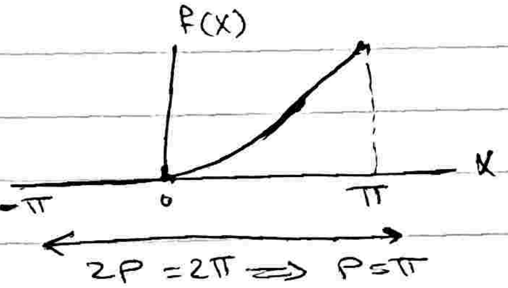
$$f(t) = \frac{\frac{2}{\pi}}{2}\Big(\frac{1}{2}\sin t\Big) + \Big(0 - \frac{2}{3\pi}\cos 2t + 0 + \cdots\Big)$$

$$\boxed{f(t) = \frac{1}{\pi} + \frac{1}{2}\sin t - \frac{2}{3\pi}\cos 2t + \cdots}$$

Ex 2 Find the Fourier series for the Following Periodic function

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x^2 & 0 < x < \pi \end{cases}$$

Sol.



$$a_0 = \frac{1}{P}\int_{b_1}^{b_2} f(x)\,dx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x^2\,dx + \cancel{\frac{1}{\pi}\int_{-\pi}^{0} 0\,dx}$$

$$a_0 = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{\pi^3}{3} - 0\right]$$

$$a_0 = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{P}\int_0^{\pi} x^2 \cos\frac{n\pi x}{P} = \frac{1}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{n}\cancel{\sin n\pi} + \frac{2\pi}{n^2}\cos n\pi - \frac{2}{n^2}\cancel{\sin n\pi}\right) - \left(0 + 0 - \frac{2}{n^3}\sin 0\right)\right]$$

$$= \frac{1}{\cancel{\pi}} + \frac{2\pi}{n^2}\cos n\pi = \frac{2}{n^2}\cos n\pi$$

dx	∫
x^2 (+)	$\cos nx$
$2x$ (−)	$\frac{1}{n}\sin nx$
2 (+)	$\frac{1}{-n^2}\cos nx$
0	$\frac{1}{n^3}\sin nx$

n	a_n
1	$a_1 = \frac{2}{1}\cos\pi = -2$
2	$a_2 = \frac{2}{4}\cos 2\pi = \frac{1}{2}$
3	$a_3 = \frac{2}{9}\cos 3\pi = -\frac{2}{9}$
4	$a_4 = \frac{2}{16}\cos 4\pi = \frac{2}{16} = \frac{1}{8}$

	+	
x^2	+	$\cos nx$
$2x$	−	$\frac{1}{n}\sin nx$
2	+	$-\frac{1}{n^2}\cos nx$
0	−	$-\frac{1}{n^3}\sin nx$

$$b_n = \frac{1}{p}\int_{b_1}^{b_2} f(x)\,\sin\frac{n\pi x}{p_{=\pi}}\,dx = \frac{1}{\pi}\int_0^{\pi} x^2 \sin nx\,dx$$

$$b_n = \frac{1}{\pi}\left[x^2 \cdot \frac{-1}{n}\cos nx + \frac{1}{n}\int 2x \cdot \cos nx\,dx\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-x^2}{n}\cos nx + \frac{2}{n}\left(x \cdot \frac{1}{n}\sin nx - \frac{1}{n}\int \sin nx\,dx\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-x^2}{n}\cos nx + \frac{2x}{n}\sin nx + \frac{2}{n^3}\cos nx\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{-\pi^2}{n}\cos n\pi + \frac{2\pi}{n^2}\cancel{\sin n\pi}^{\,0} + \frac{2}{n^3}\cos n\pi\right) - \left(0 + 0 + \frac{2}{n^3}\cos 0\right)\right]$$

$$= \frac{-\pi}{n}\cos \pi + \frac{2}{n^3\pi}\cos n\pi - \frac{2}{n^3\pi}$$

نشتق		نكامل
x^2	$+$	$\sin nx$
$2x$	$-$	$\frac{-1}{n}\cos nx$
2	$+$	$\frac{-1}{n^2}\sin nx$
0		$\frac{1}{n^3}\cos nx$

n	b_n
1	$b_1 = -\pi\cos\pi + \frac{2}{\pi}\cos\pi - \frac{2}{\pi}$ $= \pi - \frac{2}{\pi} - \frac{2}{\pi} = \boxed{\pi - \frac{4}{\pi}}$
2	$b_2 = -\frac{\pi}{2}\cos 2\pi + \frac{2}{8\pi}\cos 2\pi - \frac{2}{8\pi}$ $= \frac{-2}{\pi} + \frac{2}{8\pi} - \frac{2}{8\pi} = -\frac{\pi}{2}$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x$$

$$= \frac{\pi^2}{6} + \left(-2\cos x + \frac{1}{2}\cos 2x + \cdots\right) + \left(\left(\pi - \frac{\pi}{4}\right)\cos x + \left(\frac{-\pi}{2}\right)\cos 2x + \cdots\right)$$

⊛ Even and odd periodic function :—

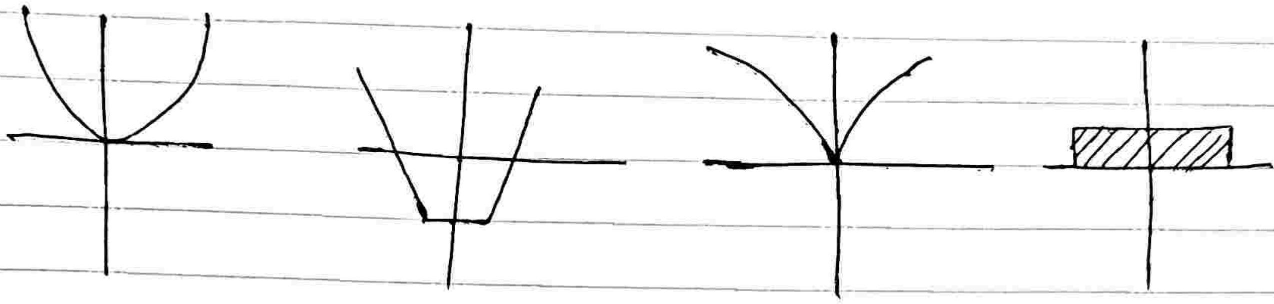
a- Even function :- A function $f(x)$ is said to be even function of x if $f(-x) = f(x)$

For examples: $f(x) = x^2$ is an even function

where $f(-x) = (-x^2) = x^2$

Note :- Even function are Symmetric with y-axis

الدالة عند رسمها تكون متناظرة حول محور (y) كما في الأمثلة التالية :-



ملاحظة في الدالة الزوجية (Even)

$a_0 = ?$

$a_n = ?$

$b_n = 0$

$$a_0 = \frac{2}{P} \int_0^P f(t)\, dt$$

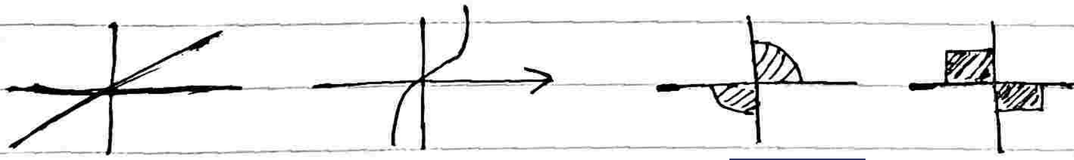
$$a_n = \frac{2}{P} \int_0^P f(t) \cos \frac{n\pi t}{P} . dt$$

b. Odd function :- A function $f(x)$ is said to be odd function of x if $f(-x) = -f(x)$

For example : $f(x) = x^3$ is an odd function, where $f(-x) = (-x)^3 = -x^3$

Note: odd function are Symmetric about (zero).

الدالة عند رسمها تكون متناظرة حول نقطة الاصل (صفر) كما في الامثلة :-



ملاحظة / في الدوال الفردية (odd)

$\boxed{a_0 = a_n = 0}$, $b_n = ?$

$$b_n = \frac{2}{P} \int_0^P f(t) \sin \frac{n\pi t}{P} . dt \Rightarrow$$ حفظ

Ex: Find the Fourier Series for the following periodic function

$f(t) = |t| \qquad -1 \leq t \leq 1$

Sol: $|t| = \begin{bmatrix} t \\ -t \end{bmatrix} \Rightarrow f(t) = \begin{bmatrix} t & 0 \leq t \leq 1 \\ -t & -1 \leq t < 0 \end{bmatrix}$

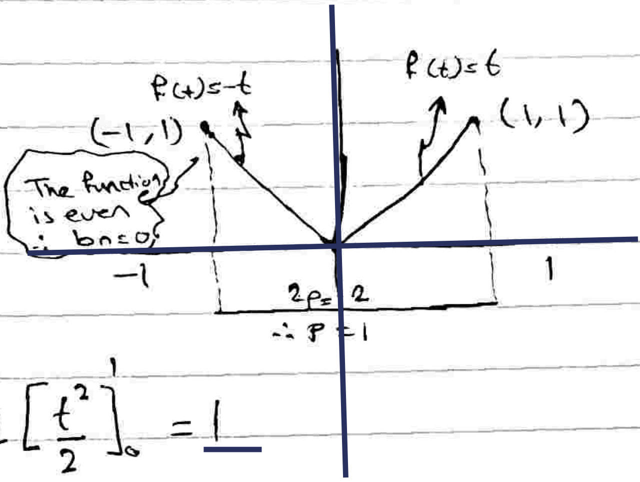
$$\int_0^1 t + \int_{-1} -t$$

$f(t) = t$

t	$f(t)$	
0	0	$\Rightarrow (0,0)$
1	1	$\Rightarrow (1,1)$

$f(t) = -t$

t	$f(t)$
0	0
-1	1



$$a_0 = \frac{2}{p}\int_0^p f(t)\,dt = \frac{2}{1}\int_0^1 t\,dt = 2\left[\frac{t^2}{2}\right]_0^1 = \underline{1}$$

$$a_n = \frac{2}{p}\int_0^p f(t)\cos\frac{n\pi t}{p}\,dt = \frac{2}{1}\int_0^1 t\cos\frac{n\pi t}{1}\,dt$$

$$= 2\left[\frac{t}{n\pi}\sin n\pi t - \int\frac{1}{n\pi}\sin n\pi t\,dt\right]$$

$$= 2\left[\frac{t}{n\pi}\sin n\pi t + \frac{1}{n^2\pi^2}\cos n\pi t\right]_0^1$$

$$= 2\left[\left(\frac{1}{n\pi}\cancel{\sin n\pi}^{\,0} + \frac{1}{n^2\pi^2}\cos n\pi\right) - \left(0 + \frac{1}{n^2\pi^2}\cancel{\cos 0}\right)\right]$$

$$\begin{array}{c|c|l} t & + & \cos n\pi t \\ 1 & - & \frac{1}{n\pi}\sin n\pi t \\ 0 & & \frac{-1}{n^2\pi^2}\cos n\pi t \end{array}$$

$$a_n = 2\left[\frac{1}{n^2\pi^2}\cos n\pi - \frac{1}{n^2\pi^2}\right] = \frac{2}{n^2\pi^2}(\cos n\pi - 1)$$

n	a_n
1	$a_1 = \frac{2}{(1)^2\pi^2}(\cos\pi - 1) = \frac{-4}{\pi^2}$
2	$a_2 = \frac{2}{2^2\pi^2}(\cos 2\pi - 1) = 0$
3	$a_3 = \frac{2}{3^2\pi^2}(\cos 3\pi - 1) = \frac{-4}{9\pi^2}$

$$\therefore f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{p} + t$$

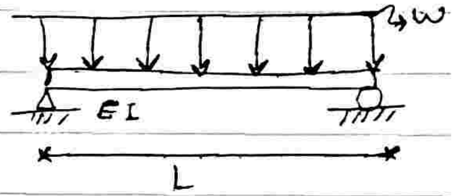
$$f(t) = \frac{1}{2} + \left(\frac{-4}{\pi^2}\cos\pi t + 0 - \frac{4}{9\pi^2}\cos 3\pi t + \underline{\cdots}\right)$$

9- Application of Fourrier Series:-

Deflection of a simply supported beam:-

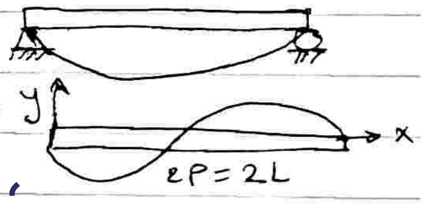
Ex-1 A simply supported beam of length (L) and uniform rigidity (EI), carried a load of (w) kN, If the self weight is neglected, find the deflection curve of the beam?

The well known relation between deflection & the applied load equation will be used to estimate the equation of deflection curve of the beam.



$$EI\, y^{IV} = (q(x)) = w$$

It is clear from the support case that the deflection curve will be compatible with half sine wave. Hence it can be represented using half sine expansion.



y = defl.
y' = slope
y'' = moment
y''' = shear
y^{IV} = load

$$y^{IV} = \frac{q(x)}{EI}$$

$$y = \sum_{n=1}^{\infty} y_n \sin\frac{n\pi x}{P}$$

$$y = \sum_{n=1}^{\infty} y_n \sin\frac{n\pi x}{L}$$ and the fourth derivatives will be:

$$y' = \frac{\pi}{L}\sum_{n=1}^{\infty} n \cdot y_n \cos\frac{n\pi x}{L}$$

$$y'' = -\frac{\pi^2}{L^2}\sum_{n=1}^{\infty} n^2 \cdot y_n \sin\frac{n\pi x}{L}$$

$$y^{IV} = \frac{\pi^4}{L^4}\sum_{n=1}^{\infty} n^4 \cdot y_n \sin\frac{n\pi x}{L}$$

The step applied load can be represented by:

$$q(x) = w$$

The applied load will be repeated to perform an odd periodic function for which the half expansion will made.

$$q(x) = \sum_{n=1}^{\infty} q_n \sin \frac{n\pi x}{P}$$

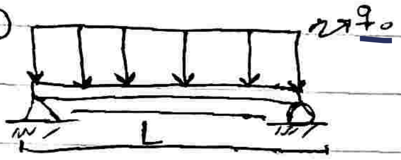
$$q_n = \frac{2}{P} \int_0^P f(x) \sin \frac{n\pi x}{P} \, dx$$

يعتمد على نوع الحمل وشكله مثلاً w

Procedure of Solution

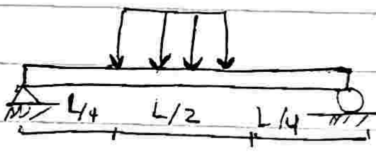
1- Find the function $q(x) = ??$

a)



$q(x) = q_0 \quad , 0 \leq x \leq L$

b)

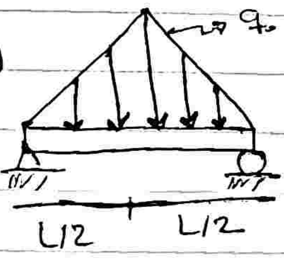


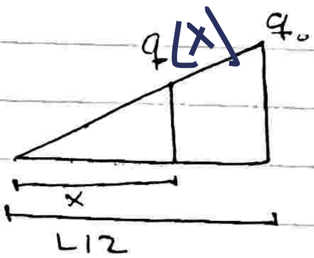
$q(x) = 0 \; , 0 \leq x \leq \frac{L}{4}$

$q_0 \; , \frac{L}{4} \leq x \leq \frac{3}{4}L$

$0 \; , 3L/4 \leq x \leq L$

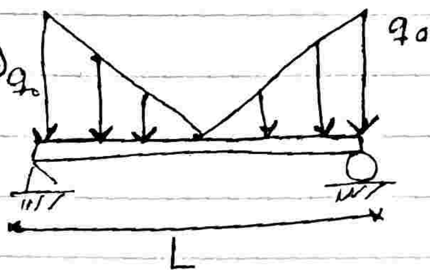
C)





$$q(x) = \begin{cases} 2 q_0 \frac{x}{L} \; , & 0 \leq x \leq \frac{L}{2} \\ 2 q_0 \left(1 - \frac{x}{L}\right) , & \frac{L}{2} \leq x \leq L \end{cases}$$

d)



$q(x) = 4 q_0 \cdot \left(\frac{x}{L} - \frac{1}{2}\right)^2 \quad 0 \leq x \leq L$

(3) Find the Fourier Series for $q(x)$

$$q(x) = \sum_{n=1}^{\infty} q_n \sin \frac{n\pi x}{L} \quad , \text{where :}$$

$$q_n = \frac{2}{L} \int_0^L q(x) \sin \frac{n\pi x}{L} \, dx$$

③ Find $y_n = ?$ for the equation of deflection by using governing equation

$$y(x) = \sum_{n=1}^{\infty} y_n \sin\frac{n\pi x}{L}$$

$$\frac{d^4y}{dx^4} = \sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^4 y_n \sin\frac{n\pi x}{L}$$

④ $\because \dfrac{d^4y}{dx^4} = \dfrac{q(x)}{EI}$, $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^4 y_n \cancel{\sin\frac{n\pi x}{L}} = \frac{\sum_{n=1} q_n \cancel{\sin\frac{n\pi x}{L}}}{EI}$

$$\therefore y_n = \left(\frac{n\pi}{L}\right)^4 * \frac{q_n}{EI}$$

⑤ Write deflection equation

$$\because y(x) = \sum_{n=1}^{\infty} y_n \sin\frac{n\pi x}{L}$$

$$\therefore y_1 * \sin\frac{\pi x}{L} + y_2 * \sin\frac{2\pi x}{L} + \cdots$$

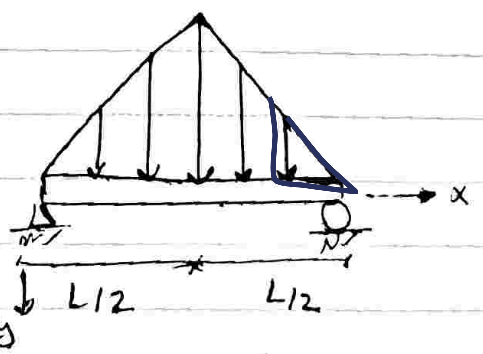
⑥ Answer any required about elastic curve.

y' , y'' , y''' → shear

y' → slope, y'' → moment

Ex2 For a simple supported beam of length (L) shown in fig. below use half range function of fourrier.

1) Max. deflection
2) Max. moment
3) Max. Rotation



Sol. 1- Find $q(x)$

$$q(x) = \begin{cases} 2q_o\dfrac{x}{L}, & 0 \le x \le \dfrac{L}{2} \\ 2q_o\left(1-\dfrac{x}{L}\right), & \dfrac{L}{2} \le x \le L \end{cases}$$

2- $q(x) = \displaystyle\sum_{n=1}^{\infty} q_n \sin\frac{n\pi x}{L}$

$f(x) \cdot \cancel{\dfrac{L}{2}} = \dfrac{2q x}{L}$

$$q_n = \frac{2}{L}\int_0^L q(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\left[\int_0^{L/2} 2q_0\frac{x}{L}\cdot\sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} 2q_0\left(1-\frac{x}{L}\right)\sin\frac{n\pi x}{L}\,dx\right]$$

u	dv		u	dv
x	$\sin\frac{n\pi x}{L}$		$1-\frac{x}{L}$	$\sin\frac{n\pi x}{L}$
1 (+)	$-\frac{L}{n\pi}\cos\frac{n\pi x}{L}$	+	$-\frac{1}{L}$ (+)	$-\frac{L}{n\pi}\cos\frac{n\pi x}{L}$
0 (−)	$-\frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}$		0 (−)	$-\frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}$

$$q_n = \frac{4q_0}{L^2}\left[\frac{-xL}{n\pi}\cos\frac{n\pi x}{L} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right]_0^{L/2} + \frac{4q_0}{L}\left[\frac{L(\frac{x}{L}-1)}{n\pi}\cos\frac{n\pi x}{L} - \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right]_{L/2}^{L}$$

$$q_n = \frac{4q_0}{l}\left(\frac{2l}{n^2\pi^2}\sin\frac{n\pi}{2}\right) = \frac{8q_0}{n^2\pi^2}\sin\frac{n\pi}{2}$$

③ find $y_n = ?$

$$\because \overset{IV}{y} = \frac{q(x)}{EI}$$

$$\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^4 y_n \sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty}\frac{8q_0}{n^2\pi^2 EI}\sin\frac{n\pi}{2}\cdot\sin\frac{n\pi x}{L}$$

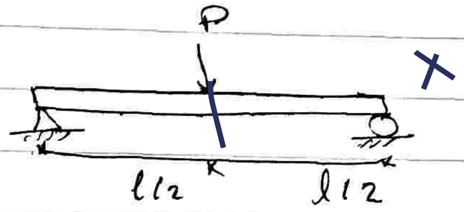
$$y_n = \frac{8q_0L^4}{n^2\pi^2 EI}\sin\frac{n\pi}{2}$$

$$\therefore y(x) = \sum_{n=1}^{\infty} y_n\sin\frac{n\pi x}{L}$$

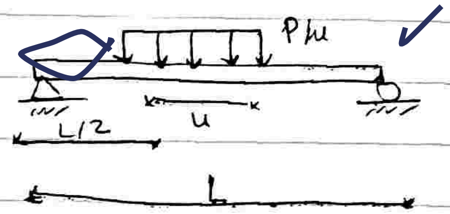
$$y(x)\ \sum_{n=1}^{\infty}\frac{8q_0 l^4}{n^2\pi^2 EI}\sin\frac{n\pi}{2}*\sin\frac{n\pi x}{2}$$

$$y(x) = \frac{8q_0 l^4}{\pi^2 EI}\left(\sin\frac{\pi x}{L} - \frac{1}{9}\sin\frac{2\pi x}{L} + \frac{1}{25}\sin\frac{3\pi x}{L}\right)$$

Ex.3 Find the deflected curve of a simply supported beam of length (L) & uniform rigidity (EI) under the effect of concentrated load (P kN) at it's mid span?



Sol. ملاحظة لا يمكن الحل الا بوجود حمل منتشر ربما .
انه يوجد حمل مركز سيفرض انه حمل منتشر على مسافة (u)
التي تكون مساوية تقريبا للصفر .



$P = 1$

$u \to 0$ صفر

$$q(x) = \begin{cases} 0, & 0 \le x \le L/2 \\ = P/u, & L/2 \le x \le (L/2 + u) \\ 0, & (L/2 + u) \le x \le L \end{cases}$$

$$\therefore q(x) = \sum_{n=1}^{\infty} q_n \sin \frac{n\pi x}{L}$$

$$q_n = \frac{2}{L} \int_0^{2} q(x) \sin \frac{n\pi x}{L}\, dx$$

$$q_n = \frac{2}{L}\left[\int_0^{L/2} 0 \sin\frac{n x\pi}{L} dx + \int_{L/2}^{L/2+u} P/u \sin\frac{n x\pi}{L} dx + \int_{L/2+u}^{0} 0 \sin\frac{n\pi x}{L} dx\right]$$

$$q_n = \frac{2}{L}\left[\int_0^{L/2} P/u \sin\frac{n\pi x}{L} dx\right.$$

$$q_n = \frac{2}{L} \int_0^{L/2} P/u \sin\frac{n\pi x}{L} dx$$

$$q_n = \frac{2P}{Lu}\left[\frac{-L}{n\pi} \cos\frac{n\pi x}{L}\right]_{l/2}^{l/2+u}$$

$$q_n = -\frac{2P}{Lu}\left[\frac{L}{n\pi} \cos\left(\frac{n\pi(\frac{l}{2}+u)}{L}\right) - \frac{L}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right]$$

$$q_n = \frac{-2P}{Lu}\left[\frac{L}{n\pi}\left(\cos\frac{n\pi}{2} \cdot \cos\frac{n\pi u}{L} - \sin\frac{n\pi}{2}\sin\frac{n\pi u}{L}\right)\right.$$

$u \to 0 \to 1$

$$q_n = \frac{-2P}{n\pi u}\left(\cos\frac{n\pi}{2} \cdot \cancel{\cos\frac{n\pi u}{1}} - \cancel{\sin\frac{n\pi}{2}} \cdot \sin\frac{n\pi u}{L} - \cancel{\cos\frac{n\pi}{2}}\right.$$

$$q_n = \frac{2P}{n\pi u} \cdot \left(\sin\frac{n\pi}{2}\sin\frac{n\pi u}{L}\right)$$

This will gives :-