FINITE DIFFERENCE

Function of one variable

Ordinary (and also partial) differential equations of order two or more conditions for complete solution.

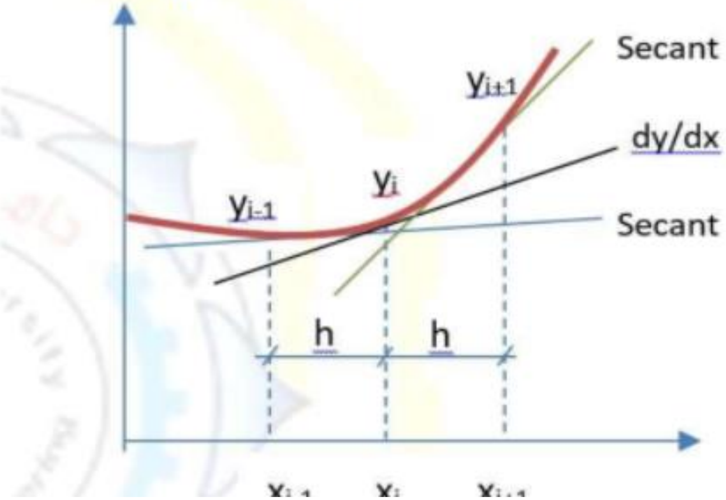
If these conditions are given at the beginning and at the end of the domain of the problem, then boundary value problem is obtained.

Numerical Solution Of Boundary Value Problems By Finite Differences

Let $y = y(x)$. The derivatives of y (such as $y' = {}^{dy}/_{dx}$, $y'' = {}^{d^2y}/_{dx^2}$, ...) are converted into expressions by ***finite differences***

Define $y_i = y$ ***at*** x_i ***(node*** i***)***

$h = \Delta x = x_{i+1} - x_i = x_{i+2} - x_{i+1}$



خطوات الحل:

- نجد قيمة h ومنها نطبق ارقام القالب الخاص بالمشتقة للسؤال
- نجد الشروط الحقيقة والثابتة والمحيطة بطبيعة المسألة

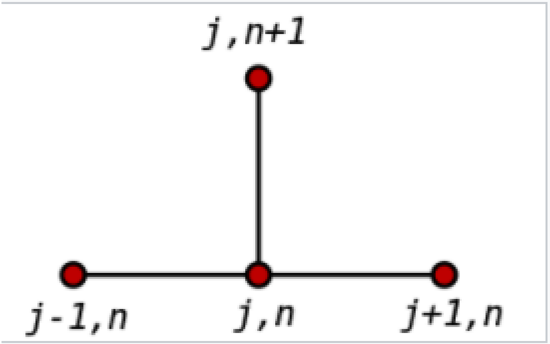
FINITE DIFFERENCE

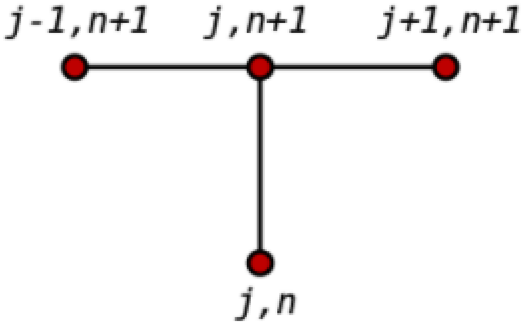
In numerical analysis, **finite-difference methods** (**FDM**) are a class of numerical techniques for solving differential equations by approximating derivatives with finite differences. Both the spatial domain and time interval (if applicable) are discretized, or broken into a finite number of steps, and the value of the solution at these discrete points is approximated by solving algebraic equations containing finite differences and values from nearby points.

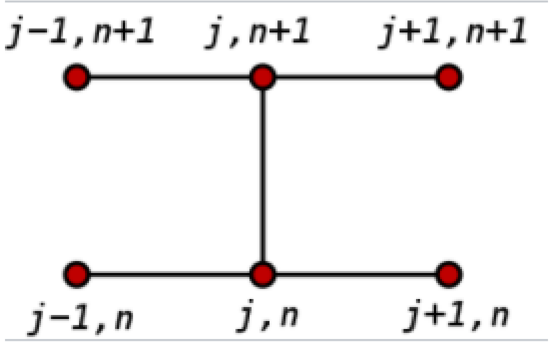
Finite difference methods convert ordinary differential equations (ODE) or partial differential equations (PDE), which may be nonlinear, into a system of linear equations that can be solved by matrix algebra techniques. Modern computers can perform these linear algebra computations efficiently which, along with their relative ease of implementation, has led to the widespread use of FDM in modern numerical analysis.[1] Today, FDM are one of the most common approaches to the numerical solution of PDE, along with finite element methods.[1]

solving differential equations by approximating derivatives

FINITE DIFFERENCE







Boundary Value Problem

1. First Derivative

- **Forward And Backward Difference**

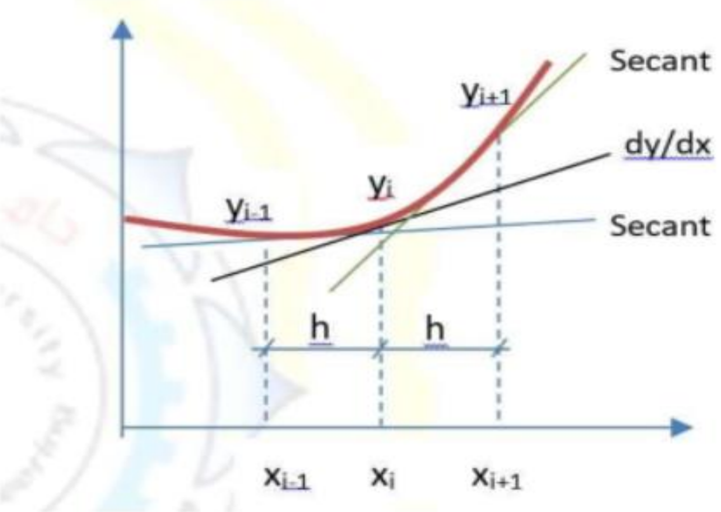
at node i

$$\frac{dy}{dx} \cong \Delta y_i = \frac{1}{h} \cdot (y_{i+1} - y_i) \qquad forward\ difference\ (\Delta)$$

$$= \nabla y_i$$

$$= \frac{1}{h} \cdot (y_i - y_{i-1}) \qquad backward\ difference\ (\nabla)$$

B. W. D < Exact Solution < F. W. D



- **Central Derivative**

at node i

$$Central\ derivative = \frac{1}{2}\ (F.W.D + B.W.D)$$

$$\frac{dy}{dx} = \frac{1}{2h} \cdot (y_{i+1} - y_{i-1})$$

OR $\left.\frac{dy}{dx}\right)_i = \frac{1}{2h}$ [(-1) — (0) — (1)]

Central difference is preferable because it has lower error.

Boundary Value Problem

2. Second Derivative

at node i

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{(dy/dx)_{i+1} - (dy/dx)_i}{h}$$ *forward* بالنسبة $\frac{d^2y}{dx^2}$

$$= \frac{(y_{i+1} - y_i)/h - (y_i - y_{i-1})/h}{h}$$ *backward* بالنسبة $\frac{dy}{dx}$

$$= \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1})$$ *central*

OR $\left.\frac{d^2y}{dx^2}\right)_i = \frac{1}{h^2}$ [(1) — (-2) — (1)]

Asst. Lect. Haider Qais
MUC 2021-2022

Brief of Pattern

1- First derivative

OR $\frac{dy}{dx})_i = \frac{1}{2h}[\ (-1) — (0) — (1)\ ]$

2- Second derivative

OR $\frac{d^2y}{dx^2})_i = \frac{1}{h^2}[\ (1) — (-2) — (1)\ ]$

3. Third derivative

$\frac{d^3y}{dx^3})_i = \frac{1}{2h^3}[\ (-1) — (2) — (0) — (-2) — (1)\ ]$

4. Forth derivative

$\frac{d^4y}{dx^4})_i = \frac{1}{h^4}[\ (1) — (-4) — (6) — (-4) — (1)\ ]$

***EX* :- The deflection equation of abeam is**

$$\frac{d^4y}{dx^4} = \frac{q(x)}{EI}$$

Where

$y = y_{(x)}$: is the deflection

q= (per unit length) is the distributed load on beam.

EI: flexural rigidity of the beam.

Consider abeam of length (L) fixed at one end and simply supported at the other end. The load on the beam is triangular as shown.

By finite differences, obtain the deflections in the beam,

Take $h = \Delta x = l/4$

***at node 0**

$$\bar{y}_0 = \frac{y_1 - y_{-1}}{2h} = 0 \quad \Rightarrow \boxed{y_{-1} = y_1}$$

***at node 4**

$$\bar{\bar{y}}_4 = \frac{1}{h^2}[y_5 - 2y_4 + y_3]$$

$$\bar{\bar{y}}_4 = 0 \quad and \quad y_4 = 0 \Rightarrow \boxed{y_5 = -y_3}$$

Here

$$\frac{d^4y}{dx^4} = \frac{q_o \cdot x/l}{EI} = \frac{q_o}{LEI}x$$

In finite differences of i :-

$$\frac{1}{h^4}[y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2}] = \frac{q_o}{LEI}xi$$

Or

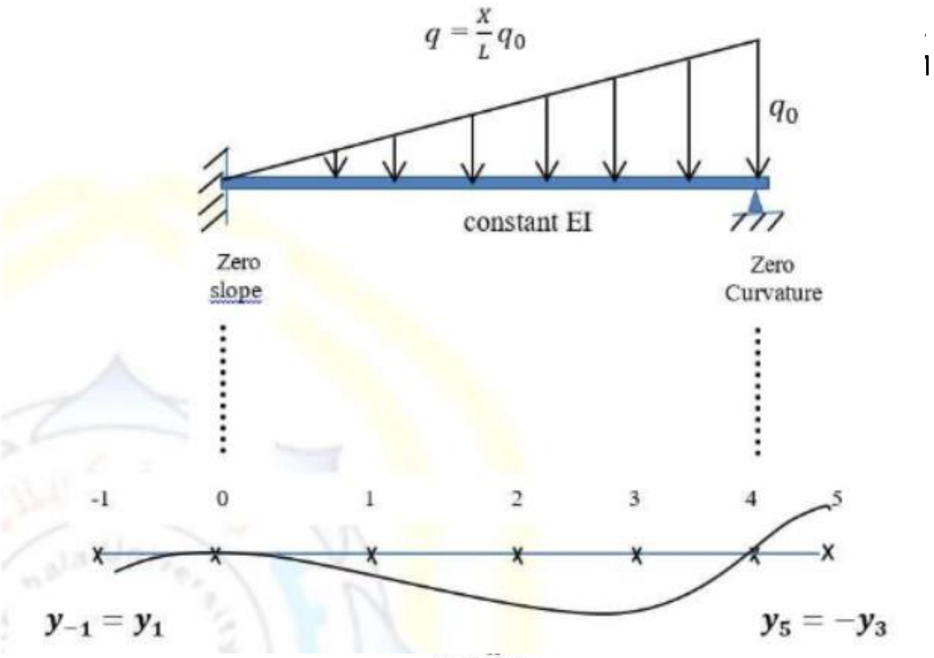
$$\frac{1}{h^4}[\ (1)-(-4)-(6)-(-4)-(1)\ ] = \frac{q_o}{LEI}xi$$

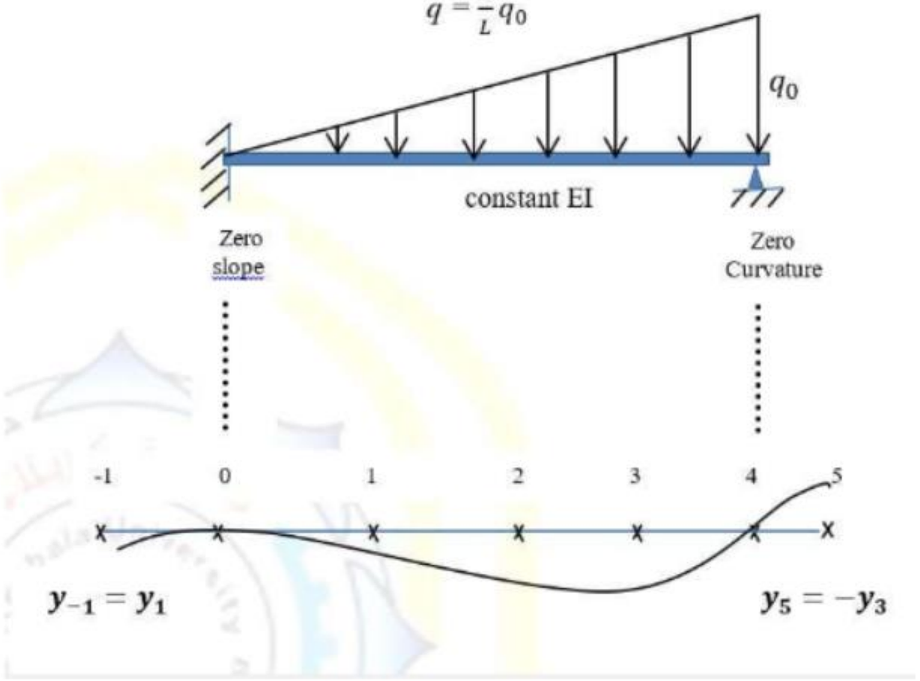
Apply to nodes 1,2 and 3 as follows :-

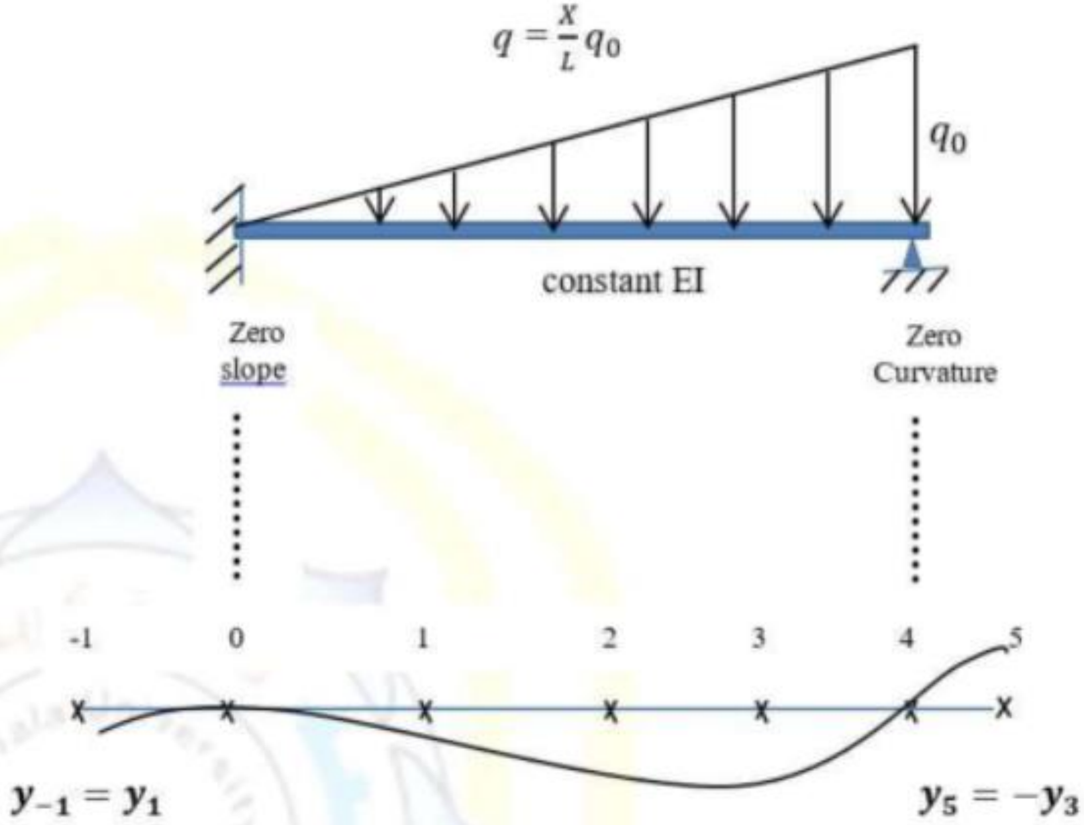
***at node 1**

$$\frac{1}{(L/4)^4}[y_{-1} - 4y_0 + 6y_1 - 4y_2 + y_3] = \frac{q_o}{LEI} \cdot L/4$$

$$\Rightarrow 7y_1 - 4y_2 + y_3 = \frac{1}{1024} \cdot \frac{q_o L^4}{EI} = A \ldots\ldots\ldots (1)$$







at node 2

$$\frac{1}{(L/4)^4}[y_0 - 4y_1 + 6y_2 - 4y_3 + y_4] = \frac{q_o}{LEI} \cdot {}^{L}/_{2}$$

$$\Rightarrow -4y_1 + 6y_2 - 4y_3 = 2A \dots\dots\dots (2)$$

at node 3

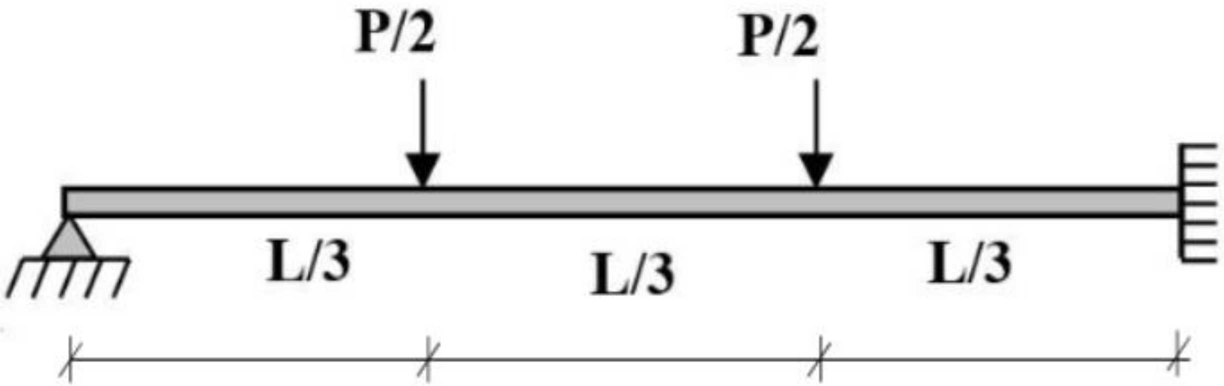
$$\frac{1}{(L/4)^4}[y_1 - 4y_2 + 6y_3 - 4y_4 + y_5] = \frac{q_o}{LEI} \cdot {}^{3L}/_{4}$$

$$\Rightarrow y_1 - 4y_2 + 5y_3 = 3A \dots\dots\dots (3)$$

Solve these three equations

$y_1 = 1.6868\times10^{-3}\,\frac{q_oL^4}{EI}$, $y_2 = 3.4624 \times 10^{-3}\,\frac{q_oL^4}{EI}$, $y_3 = 3.01847 \times 10^{-3}\,\frac{q_oL^4}{EI}$

Ex2: Find ultimate mid-span deflecti
? Use F.D.M with h= L/4



$$\frac{d^4y}{dx^4} = \frac{q(x)}{EI}$$

In finite differences of i :-

$$\frac{1}{h^4}[y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2}] = \frac{q_{(X)}}{EI}$$

Or

$$\frac{1}{h^4}[\ (1) - (-4) - (6) - (-4) - (1)\ ] = \frac{q_{(X)}}{EI}$$

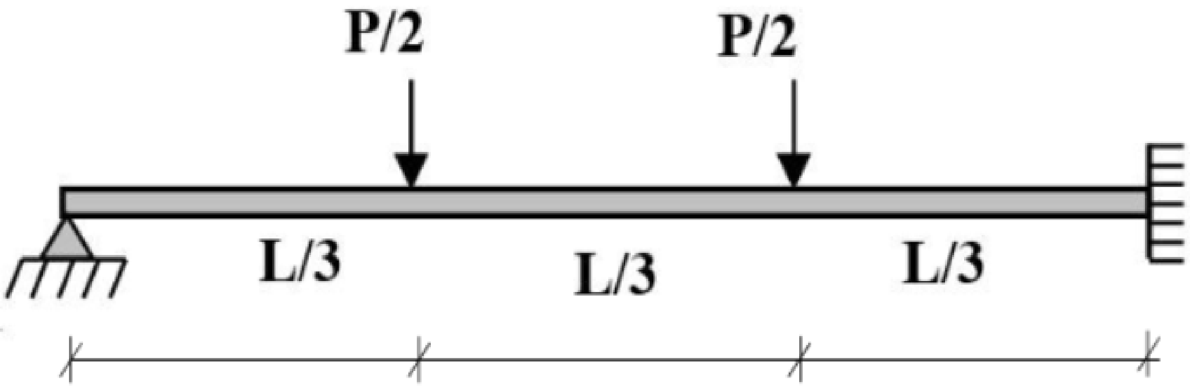
Apply to nodes 1,2 and 3 as follows :-

****at node 1***

$$q1(x) = concentrated\ load/h = (\frac{p}{2})/h$$

$$\frac{1}{(L/4)^4}[y_{-1} - 4y_0 + 6y_1 - 4y_2 + y_3] = \frac{p/2}{(\frac{L}{4})EI}$$

$$\Rightarrow 5y_1 - 4y_2 + y_3 = \frac{1}{128}.\frac{pl^3}{EI} = A \ldots\ldots\ldots(1)$$



at node 2

$$q1(x) = 0$$

$$\frac{1}{(L/4)^4}[y_0 - 4y_1 + 6y_2 - 4y_3 + y_4] = 0$$

$$\Rightarrow -4y_1 + 6y_2 - 4y_3 = 0 \ldots\ldots\ldots(2)$$

at node 3

$$q1(x) = concentrated\ load/h = (\frac{p}{2})/h$$

$$\frac{1}{(L/4)^4}[y_1 - 4y_2 + 6y_3 - 4y_4 + y_5] = \frac{p/2}{(\frac{L}{4})EI}$$

$$\Rightarrow y_1 - 4y_2 + 7y_3 = A \ldots\ldots\ldots(3)$$

Solve these three equations

Numerical integrations

Numerical Integration.

1-The Method Of Trapezoids   2-Simpson's 1/3 Rule   3- Method of Undetermined coefficients

4- Gaussian Quadrature.

$$I = \int_{-1}^{1} f(t).dt = C_1.f(t_1) + C_2.f(t_2) + \cdots \ldots .C_n.f(t_n)$$

t_1, t_2, t_3 *optimal points*

خطوات الحل:

-نستخدم الجدول ادناه مع الانتباه الى عدد النقاط المأخوذة

-ناخذ قيمة w من الجدول وتضرب في كل حد من حدود التكامل في السؤال

- نعوض قيم x بما يساويها من الجدول

Number of Points	Locations, x_i	Associated Weights, W_i
1	$x_1 = 0.000\ldots$	2.000
2	$x_1, x_2 = \pm 0.57735026918962$	1.000
3	$x_1, x_3 = \pm 0.77459666924148$	$\frac{5}{9} = 0.555\ldots$
	$x_2 = 0.000\ldots$	$\frac{8}{9} = 0.888\ldots$
4	$x_1, x_4 = \pm 0.8611363116$	0.3478548451
	$x_2, x_3 = \pm 0.3399810436$	0.6521451549

Numerical integrations

Three point quadrature.

$$\int_{-1}^{1} f(t).dt = 0.555 * f(-0.774) + 0.888 * f(0) + 0.555 * f(0.774)$$

Number of Points	Locations, x_i	Associated Weights, W_i
1	$x_1 = 0.000\ldots$	2.000
2	$x_1, x_2 = \pm 0.57735026918962$	1.000
3	$x_1, x_3 = \pm 0.77459666924148$	$\frac{5}{9} = 0.555\ldots$
	$x_2 = 0.000\ldots$	$\frac{8}{9} = 0.888\ldots$
4	$x_1, x_4 = \pm 0.8611363116$	0.3478548451
	$x_2, x_3 = \pm 0.3399810436$	0.6521451549

Numerical integrations

Evaluate the integrals (a) $I = \int_{-1}^{1}[x^2 + \cos(x/2)]\,dx$ and (b) $I = \int_{-1}^{1}(3^x - x)dx$ using three-point Gaussian quadrature.

SOLUTION:

(a) Using Table 10–2 for the three Gauss points and weights, we have $x_1 = x_3 = \pm 0.77459\ldots$, $x_2 = 0.000\ldots$, $W_1 = W_3 = \frac{5}{9}$, and $W_2 = \frac{8}{9}$. The integral then becomes

$$I = (-0.77459)^2 + \cos\left(-\frac{0.77459}{2}\text{ rad}\right)\ \frac{5}{9} + \ 0^2 + \cos\frac{0}{2}\ \frac{8}{9}$$

$$+ \ (0.77459)^2 + \cos\left(\frac{0.77459}{2}\text{ rad}\right)\ \frac{5}{9}$$

$$= 1.918 + 0.667 = 2.585$$

Numerical integrations

(b) Using Table 10–2 for the three Gauss points and weights as in part (a), the integral then becomes

$$I = [3^{(-0.77459)} - (-0.77459)]\frac{5}{9} + [3^{0} - 0]\frac{8}{9} + [3^{(0.77459)} - (0.77459)]\frac{5}{9}$$

$$= 0.66755 + 0.88889 + 0.86065 = 2.4229(2.423 \text{ to four significant figures})$$

Compared to the exact solution, we have $I_{exact} = 2.427$. The error is $2.427 - 2.423 = 0.004$. ■

Numerical integrations

<u>خطوات الحل:</u>

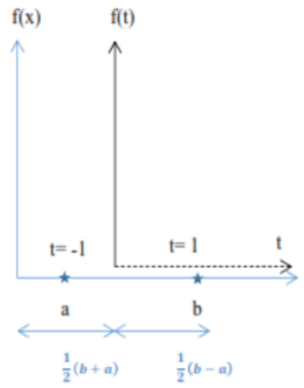
-نستخدم الجدول ادناه مع الانتباه الى عدد النقاط المأخوذة في حال كون الحدود ليست من -1 لغية 1

-ناخذ قيمة w من الجدول وتضرب في كل حد من حدود التكامل في السؤال

- نعوض قيم x بما يساويها من الجدول

Limits of integration in <u>Gaussian Quadrature:-</u>

$$x = \frac{1}{2}(b+a) + \frac{1}{2}(b-a)t \qquad dx = \frac{1}{2}(b-a)dt$$



Numerical integrations

EX:- *find the integral* $I=\int_0^1 \frac{x}{\sin x}\,.dx$

Sol:

$$x=\frac{1}{2}(b+a)+\frac{1}{2}(b-a)t=\frac{1}{2}+\frac{1}{2}t$$

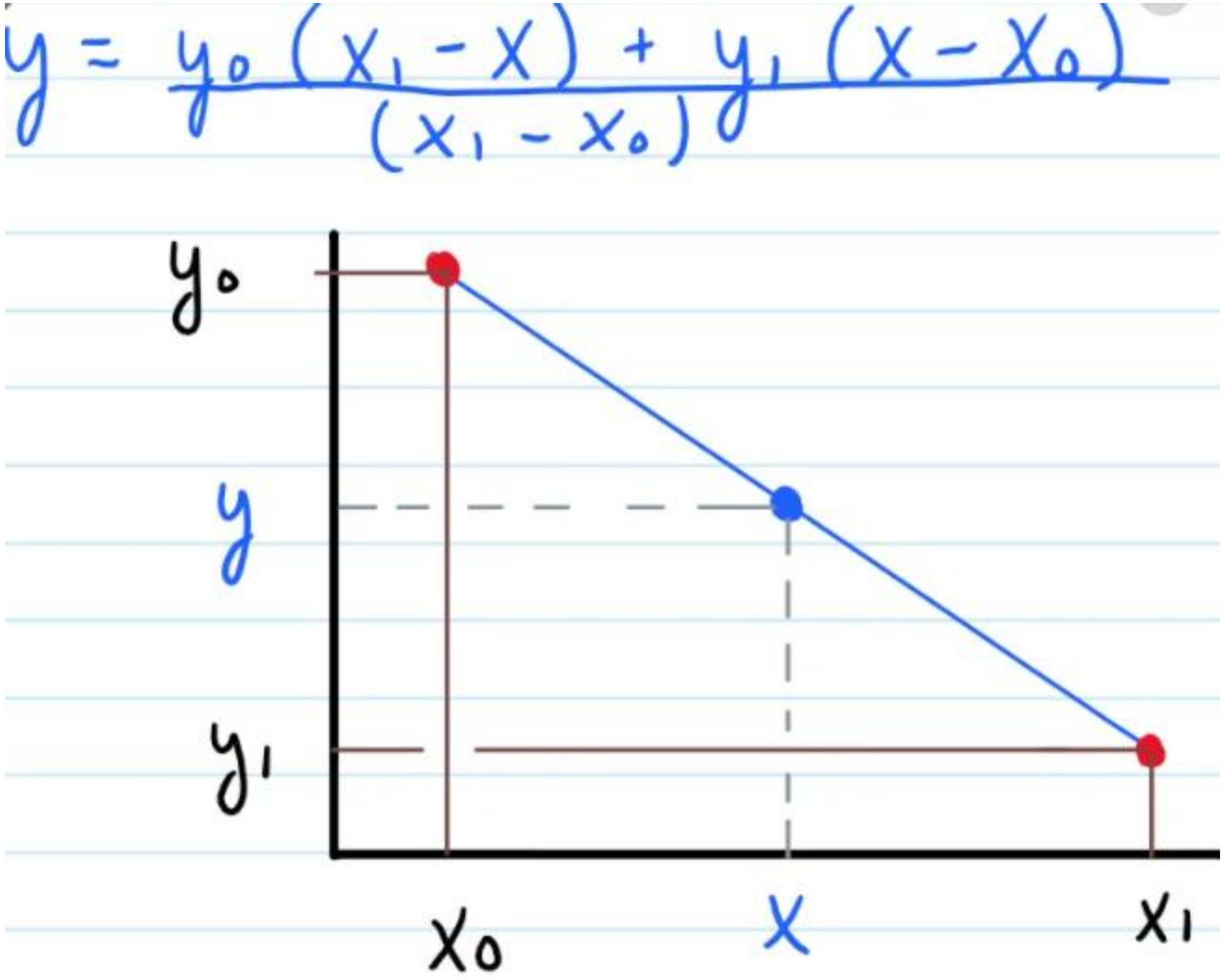
$$dx=\frac{1}{2}dt$$

$$\therefore \int_0^1 \frac{x}{\sin x}.dx=\int_{-1}^1 \frac{(\frac{1}{2}+\frac{1}{2}t)}{\sin(\frac{1}{2}+\frac{1}{2}t)}.\frac{1}{2}dt \quad = \frac{1}{4}\int_{-1}^1 \frac{(1+t)}{\sin 0.5(1+t)}.dt$$

$$=\frac{1}{4}\left[0.555*\frac{1-0.774}{\sin\left[\frac{1}{2}(1-0.774)\right]}+0.888*\frac{1+0}{\sin\left[\frac{1}{2}(1+0)\right]}+0.555*\frac{1+0.774}{\sin\left[\frac{1}{2}(1+0.774)\right]}\right]=$$

$$=\frac{1}{4}[1.1124+1.8522+1.2701]=1.059$$

LINEAR INTERPOLATION



$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

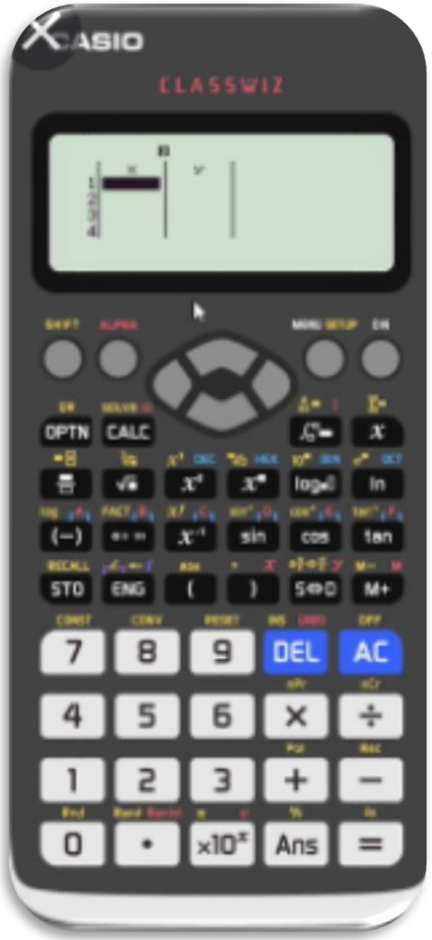
$$y = \frac{y_0(x_1 - x) + y_1(x - x_0)}{x_1 - x_0}$$

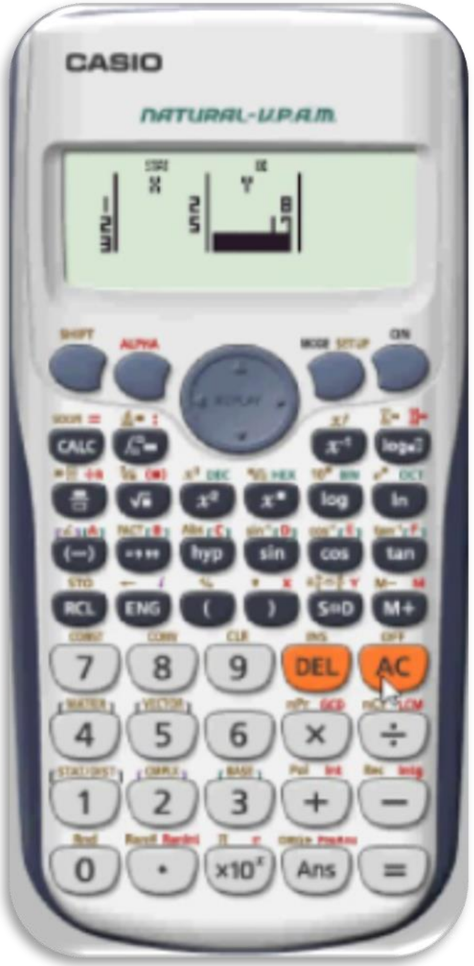
X_1	Y_1
X_2	Y_2
X_3	Y_3

$$y_2 = \frac{(x_2 - x_1)(y_3 - y_1)}{(x_3 - x_1)} + y_1$$

:خطوات الحل باستخدام الحاسبة العلمية

- Setup
- Statistics
- Y=a+bx
- Insert data for x and y (the actual)
- AC (bottom)
- Enter the required number then OPTN then Regression then Ý





Ex: fin the value of y when x=2.5

X	Y
2	6
3	8

Example:-

Using the method of proportional parts, find y at $x=0.5$, $x=0.75$, given the following table:

x	0	1	2	5
y	2	3	12	147

Solution:-

$$y_k = y_r + \left(\frac{y_{r+1} - y_r}{x_{r+1} - x_r}\right)(x_k - x_r)$$

$$y(0.5) = 2 + \left(\frac{3-2}{1-0}\right)(0.5-0) = 2.5$$

$$y(0.75) = 2 + \left(\frac{3-2}{1-0}\right)(0.75-0) = 2.75$$

Given data

x1	2	4	y1
x2	2.5	5	y2
x3	3	6	y3

$$y_2 = \frac{(x_2 - x_1)(y_3 - y_1)}{(x_3 - x_1)} + y_1$$

y2= 5

GREGORY- NEWTON FORWARD INTERPOLATION

If $y(x)$ is a polynomail of the n^{th} degree $\Delta^{n+1} y_0$ —— are Zero

Hence

$$y_n(x) = y_n(x_0+uh) = y_0 + \frac{u^{(1)}}{1!}\Delta y_0 + \frac{u^{(2)}}{2!}\Delta^2 y_0 + \cdots + \frac{u^{(n)}}{n!}\Delta^n y_0 \quad \text{-----(1)}$$

where,

$$u = \frac{x - x_0}{h}, \quad u^{(r)} = u(u-1)(u-2)\cdots(u-\overline{r-1})$$

(if x is given, u is found out)

Equation (1) is known as Gregory - Newton forward interpolation formula.

This method can be used for interpolation a value of y nearer to the end of the table values.

If y(x) is a polynomial of the nth degree hence eq (1) in Gregory Newton B.W.I becomes as below.

$$y_n(x) = y_n(x_n + vh) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \dots \frac{v(v+1)(v+2)+\dots(v+n-1)}{n!}\nabla^n y_n$$

where $v = \frac{x - x_n}{h}$

This method may also be used to interpolate closer to the right of y_n.

جمع Backward

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots + \frac{p(p+1)(p+2)\cdots(p+n-1)}{n!}\nabla^n y_n + \text{Error}$$

طرح Forward

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1)(p-n+1)}{n!}\Delta^n y_0 + \text{Error}$$

Here $p = \frac{x - x_n}{h}$

<u>خطوات الحل:</u>

- أكمال الجدول ومعرفة قيمة الزيادة في معلومات قيم x وتمثل h
- اكمال الجدول وتقدير الحل هو هو تقدمي او رجوع وبعدها ايجاد U اوP
- تطبيق القانون

Ex: find the value of y at x=21 from the following data

	X	Y	∇y	$\nabla^2 y$	$\nabla^3 y$
X0	20	0.342			
			0.0487		
X1	23	0.3907		-0.001	
			0.0477		-0.0003
X2	26	0.4384		-0.0013	
			0.0464		
X3	29	0.4848			
	h=	3			

U=(x-Xo)/h
=(21-20)/3
= 0.3333

بسبب قرب قيمة المطلب
من البداية يفضل الاخذ
Forward Interpolation

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1)(p-n+1)}{n!}\Delta^n y_0 + \text{Error}$$

Y(21) = 0.3420+(0.3333)(0.0487)+0.5*(0.3333)(0.3333-1)(-0.001)+(1/6)(0.3333(0.3333-1)(0.3333-2)(-0.0003)
Y(21) = 0.3583

GREGORY- NEWTON BACKWARD INTERPOLATION

Ex: find the value of y at x=28 from the following data

	X	Y	∇y	$\nabla^2 y$	$\nabla^3 y$
X0	20	0.342			
			0.0487		
X1	23	0.3907		-0.001	
			0.0477		-0.0003
X2	26	0.4384		-0.0013	
			0.0464		
X3	29	0.4848			
	X من السؤال	28			
	h=	3	u=	-0.33333	

U=(x-Xo)/h
=(28-29)/3
= -0.3333

بسبب قرب قيمة المطلب من النهاية يفضل الاخذ
Backward Interpolation

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$
$$+ \frac{p(p+1)(p+2)\cdots(p+n-1)}{n!}\nabla^n y_n + \text{Error}$$

$$p = \frac{x - x_n}{h}$$

Y(28) = 0.484+(-0.3333)(0.0464)+0.5*(-0.3333)(-0.3333+1)(-0.0013)+(1/6)(-0.3333(-0.3333+1)(-0.3333+2)(-0.0003)
Y(28) = 0.4695

Example: What will be the population in 1925 as per following data table

X (Year)	1891	1901	1911	1921	1931
Y(Population)	46	66	81	93	101

Solution

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

here

$\nabla Y_n = 8$ $\nabla^2 Y_n = -4$ $\nabla^3 Y_n = -1$ $\nabla^4 Y_n = -3$

$Y_n = 101$ $P = (X - X_n)/h = (1925-1931)/10 = \mathbf{-0.6}$

hence

f(1925)=

101 +(-0.6)(8)+ [(-0.6)(0.4)(-4)]/!2 + [(-0.6)(0.4)(1.4)(-1)]/!3 +[(-0.6)(0.4)(1.4)(2.4)(-3)]/!4

=101-4.8+[(0.96)/2]+[(0.336)/6]+[(2.4192)/24]

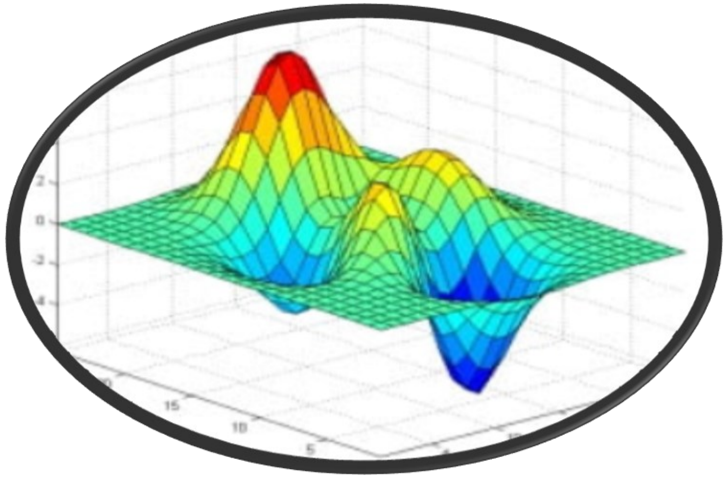
=101-4.8+0.48+0.056+0.1008

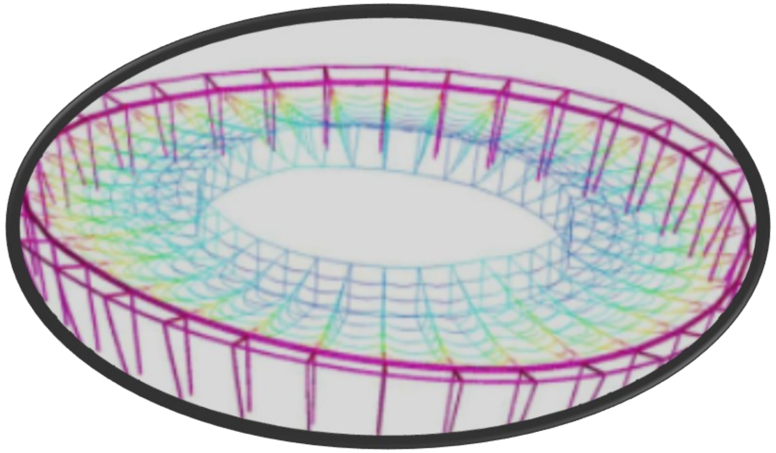
=**96.8368**

Numerical analysis course 2022/2023

Al Mansour University College

Third level





Prepared by
Asst. Lect. Haider Qais

Syllabus

- Curve fitting Interpolations
- Curve least square regression
- Numerical integration
- System of linear equations
- Solution of non linear equations
- Solution of differential equations
- Numerical solution of ordinary differential equations
- Numerical solution of partial differential equations
- Finite difference method
- Foureir series

References

- Advanced numerical and engineering analysis, wylie
- Advanced engineering mathematics", erwine

Curve fitting Interpolations

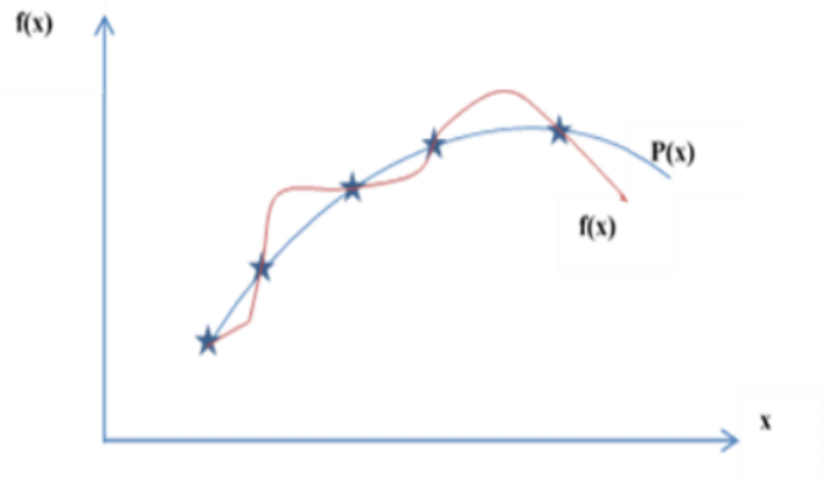
Approximation and Interpolation

Suppose that by experiments or observations, the following data are given

x_i	y_i
x_0	y_0
x_1	y_1
x_2	y_2
x_3	y_3

Required :- y_{values} *for certain* x

Polynomial interpolation



$P(x_i) = f(x_i) =$ $\quad i = 1, 2, 3, \ldots\ldots, n$

1-Newton's Divided Difference Formula

x_0	x_1	x_2
$f(x_0)$	$f(x_1)$	$f(x_2)$

$F(x) = ?$ *for given* $x_0 \le x \le x_2$

$$f(x, x_0, x_1, x_2) = \frac{f(x, x_0, x_1) - f(x_0, x_1, x_2)}{x - x_2} \quad .....(1)$$

$$f(x, x_0, x_1,) = \frac{f(x, x_0) - f(x_0, x_1)}{x - x_1} \quad .....(2)$$

$$f(x, x_1,) = \frac{f(x) - f(x_0)}{x - x_0} \quad .....(3)$$

From above equations

$$F(x) = f(x_0) + (x - x_0).f(x_0, x_1) + (x - x_0).(x - x_1).f(x_0, x_1, x_2) + Error$$

خطوات الحل:
-نكمل الجدول
-نطبق القانون

Ex1: Fit the following data

	x	F(0)	f(0,1)	F(0,1,2)	f(0,1,2,3)	f(0,1,2,3,4)
X0	-1	13	1	-1	1	0
X1	1	15	-2	4	1	
X2	2	13	10	8		
X3	4	33	34			
X4	5	67				

$$F(x) = f(x_0) + (x - x_0).f(x_0, x_1) + (x - x_0).(x - x_1).f(x_0, x_1, x_2) + (x - x_0).(x - x_1).(x - x_2).f(x_0, x_1, x_2, x_3) + Error$$

F(x)=13+[(x-(-1))*(1)]+[(x-(-1))*(x-1)*(-1)]+[(x-(-1))*(x-1)*(x-2)*(1)]+0

X^2-1

F(x) =13+[x+1]+[(x+1)(x-1)(-1)]+[(x+1)(x-1)(x-2)]

$F(x) = 13 + x + 1 - x^2 + 1 + x^3 - 2x^2 - x + 2$

$F(x) = 17 - 3x^2 + X^3$

X	F(x)	1st	2nd	3rd	4th
-1	13				
		$\frac{13-15}{-1-1}=1$			
1	15		$\frac{1+2}{-1-2}=-1$		
		$\frac{15-13}{1-2}=-2$		$\frac{-1-4}{-1-4}=1$	
2	13		$\frac{-2-10}{1-4}=4$		$\frac{1-1}{-1-5}=0$
		$\frac{13-33}{2-4}=10$		$\frac{4-8}{1-5}=1$	
4	33		$\frac{10-34}{2-5}=8$		
		$\frac{33-67}{4-5}=34$			
5	67				

HW: find the polynomial when X=4.5

2-Newton-Raphson method or open method

This method is commonly used because of its simplicity and great speed. If the initial guess at the root is x_o, a tangent can be extended from the point $(x_o, f(x_o))$. The point where this tangent crosses the x axis usually represents an improved estimate of the root. This method can be derived geometrically as follows (see the figure below)

$$\tan\theta = f'(x_o) = \frac{f(x_o) - 0}{x_o - x_1}$$

$$x_o - x_1 = \frac{f(x)}{f'(x)} \quad \Rightarrow \quad x_1 = x_o - \frac{f(x_o)}{f'(x_o)}$$

Or generally,

$$\boldsymbol{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},} \qquad \boldsymbol{f'(x_n) \neq 0}$$

القيمة الابتدائية

دالة السؤال

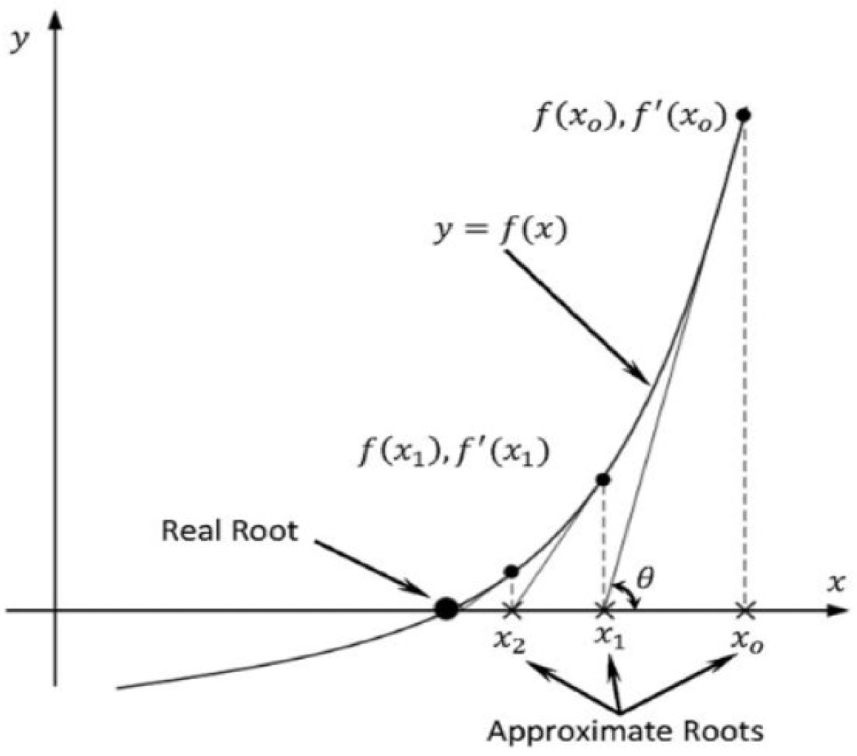
مشتقة دالة السؤال

Algorithm: Newton's Method

1. Given a function $f(x)$ real and continuous and has a continuous derivative.
2. Given a starting value x_o(initial guess).
3. Repeat the following steps until termination:
   a. Compute $f(x_n), f'(x_n)$ (if $f'(x_n) = 0$ stop, pitfall).
   b. If $f(x_n) = 0$, then the root is $x_n$and terminate the computation. Else,
   c. Compute

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

   d. Test for termination (Termination Criteria):
      i. If $|x_m^{n+1} - x_m^n| \le \epsilon$ ($\epsilon > 0$, specified tolerance)
      ii. If $|f(x_m)| \le \alpha$ ($\alpha > 0$, specified tolerance)
      iii. After N steps (N, fixed)



<u>خطوات الحل:</u>
-نساوي المعادلة الى الصفر و نشتق الدالة
-نعوض القيمة الابتدائية المفروضة
-نطبق القانون
الحاسبة العلمية
RADIAN

Ex2:

Find the positive solution of $2\sin x = x$ by using Newton's Method. (*Assume* $x_o = 2.000$ *and correct to three decimals, 3D*)

Solution

$f(x) = x - 2\sin x = 0$
$f'(x) = 1 - 2\cos x$

$$x_{n+1} = x_n - \frac{x_n - 2\sin x_n}{1 - 2\cos x_n},$$

n	x_n	x_{n+1}	$\lvert f(x_{n+1})\rvert$	$x_{n+1} - x_n$
0	2.000	1.901	0.009	0.099
1	1.901	1.896	0.000	0.005
2	1.896	1.895	0.000	0.001
3	1.895	1.895	0.000	0.000

The root is $\mathbf{1.895}$ because $|x_m^3 - x_m^4| = 0$ *and* $f(x_m^4) = 0$ ***Ans.***

The above table can be performed by using scientific calculator as follows:

Fix 0 ~ 9?	3
0.000	
2	exe
2.000	
Ans-(Ans-2sin(Ans))÷(1-2cos(Ans))	exe
1.901	
Ans-(Ans-2sin(Ans))÷(1-2cos(Ans))	exe
1.896	
Ans-(Ans-2sin(Ans))÷(1-2cos(Ans))	exe
1.895	
Ans-(Ans-2sin(Ans))÷(1-2cos(Ans))	exe
1.895	

Ex3: assuming c=2 and initial value is 1.5

Use Newton's Method to find the solution of $x = \sqrt{C}$, where C is any positive number.

to find $\sqrt{2}$. ($Assume\ x_o = 1\ and\ correct\ to\ 6D$)

Solution

Let $x^2 = C$

$f(x) = x^2 - C$

$f'(x) = 2x$

$$x_{n+1} = x_n - \frac{x_n^2 - C}{2x_n} = x_n - \frac{1}{2}\left(x_n - \frac{C}{x_n}\right) = \frac{1}{2}\left(x_n + \frac{C}{x_n}\right)$$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

n	x_{n+1}	$\lvert f(x_{n+1})\rvert$	$x_{n+1} - x_n$
0	1.500000	0.250000	0.500000
1	1.416667	0.006944	0.083333
2	1.414216	0.000006	0.002451
3	1.414214	0.000000	0.000002
4	1.414214	0.000000	0.000000

The above table can be performed by using scientific calculator as follows:

Input	Display	Key
Fix 0 ~ 9?	0.000000	6
1.5	1.500000	exe
0.5(Ans+2÷Ans)	1.416667	exe
0.5(Ans+2÷Ans)	1.414216	exe
0.5(Ans+2÷Ans)	1.414214	exe
0.5(Ans+2÷Ans)	1.414214	exe

Systems of Linear Equations:

The Gauss-Jordan Method

د. لبنى عبد الرحمن خضير

م.م. حيدر قيس

Systems of Linear Equations:

$$\begin{aligned} 3x - 2y + 8z &= 9 \\ -2x + 2y + \ \ z &= 3 \\ x + 2y - 3z &= 8 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & -2 & 8 & 9 \\ -2 & 2 & 1 & 3 \\ 1 & 2 & -3 & 8 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Row-Reduced Form of a Matrix

Each row consisting entirely of zeros lies below all rows having nonzero entries.

The first nonzero entry in each nonzero row is 1 (called a leading 1).

In any two successive (nonzero) rows, the leading 1 in the lower row lies to the right of the leading 1 in the upper row.

If a column contains a leading 1, then the other entries in that column are zeros.

Row Operation

1. Interchange any two rows.
2. Replace any row by a nonzero constant multiple of itself.
3. Replace any row by the sum of that row and a constant multiple of any other row.

Terminology for the Gauss-Jordan Elimination Method

Unit Column

A column in a coefficient matrix is in unit form if one of the entries in the column is a 1 and the other entries are zeros.

Pivoting

The sequence of row operations that transforms a given column in an augmented matrix into a unit column.

Notation for Row Operations

Letting R_i denote the ith row of a matrix, we write

Operation 1: $R_i \leftrightarrow R_j$ to mean: Interchange row i with row j.

Operation 2: cR_i to mean: replace row i with c times row i.

Operation 3: $R_i + aR_j$ to mean: Replace row i with the sum of row i and a times row j.

Example

Pivot the matrix about the circled element

$$\left[\begin{array}{cc|c} \textcircled{3} & 5 & 9 \\ 2 & 3 & 5 \end{array}\right]$$

Solution

$$\left[\begin{array}{cc|c} 3 & 5 & 9 \\ 2 & 3 & 5 \end{array}\right] \tfrac{1}{3}R_1 \Rightarrow \left[\begin{array}{cc|c} 1 & \tfrac{5}{3} & 3 \\ 2 & 3 & 5 \end{array}\right] R_2 - 2R_1 \Rightarrow \left[\begin{array}{cc|c} 1 & \tfrac{5}{3} & 3 \\ 0 & -\tfrac{1}{3} & -1 \end{array}\right]$$

The Gauss-Jordan Elimination Method

1. Write the augmented matrix corresponding to the linear system.
2. Interchange rows, if necessary, to obtain an augmented matrix in which the first entry in the first row is nonzero. Then pivot the matrix about this entry.
3. Interchange the second row with any row below it, if necessary, to obtain an augmented matrix in which the second entry in the second row is nonzero. Pivot the matrix about this entry.
4. Continue until the final matrix is in row-reduced form.

Example

Use the Gauss-Jordan elimination method to solve the system of equations

$$
\begin{aligned}
3x - 2y + 8z &= 9 \\
-2x + 2y + z &= 3 \\
x + 2y - 3z &= 8
\end{aligned}
$$

Solution

$$
\left[\begin{array}{ccc|c} 3 & -2 & 8 & 9 \\ -2 & 2 & 1 & 3 \\ 1 & 2 & -3 & 8 \end{array}\right] R_1 + R_2
$$

Toggle slides back and forth to compare before and after matrix changes

Example

Use the Gauss-Jordan elimination method to solve the system of equations

$$\begin{aligned} 3x - 2y + 8z &= 9 \\ -2x + 2y + z &= 3 \\ x + 2y - 3z &= 8 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 0 & 9 & 12 \\ -2 & 2 & 1 & 3 \\ 1 & 2 & -3 & 8 \end{array}\right] \begin{array}{l} R_1 + R_2 \\ R_2 + 2R_1 \\ R_3 - R_1 \end{array}$$

Toggle slides back and forth to compare before and after matrix changes

Example

Use the Gauss-Jordan elimination method to solve the system of equations

$$
\begin{aligned}
3x - 2y + 8z &= 9 \\
-2x + 2y + z &= 3 \\
x + 2y - 3z &= 8
\end{aligned}
$$

Solution

$$
\left[\begin{array}{ccc|c} 1 & 0 & 9 & 12 \\ 0 & 2 & 19 & 27 \\ 0 & 2 & -12 & -4 \end{array}\right] \begin{array}{l} \\ R_2 + 2R_1 \\ R_3 - R_1 \end{array}
$$

Toggle slides back and forth to compare before and after matrix changes

Example

Use the Gauss-Jordan elimination method to solve the system of equations

$$
\begin{aligned}
3x - 2y + 8z &= 9 \\
-2x + 2y + z &= 3 \\
x + 2y - 3z &= 8
\end{aligned}
$$

Solution

$$
\left[\begin{array}{ccc|c}
1 & 0 & 9 & 12 \\
0 & 2 & -12 & -4 \\
0 & 2 & 19 & 27
\end{array}\right] \quad \tfrac{1}{2}R_2
$$

Toggle slides back and forth to compare before and after matrix changes

Example

Use the Gauss-Jordan elimination method to solve the system of equations

$$
\begin{aligned}
3x - 2y + 8z &= 9 \\
-2x + 2y + z &= 3 \\
x + 2y - 3z &= 8
\end{aligned}
$$

<u>Solution</u>

$$
\left[\begin{array}{ccc|c} 1 & 0 & 9 & 12 \\ 0 & 1 & -6 & -2 \\ 0 & 2 & 19 & 27 \end{array}\right] \quad \begin{array}{l} \\ \tfrac{1}{2}R_2 \\ R_3 - R_2 \end{array}
$$

Toggle slides back and forth to compare before and after matrix changes

Example

Use the Gauss-Jordan elimination method to solve the system of equations

$$\begin{aligned} 3x - 2y + 8z &= 9 \\ -2x + 2y + z &= 3 \\ x + 2y - 3z &= 8 \end{aligned}$$

<u>Solution</u>

$$\left[\begin{array}{ccc|c} 1 & 0 & 9 & 12 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & 31 & 31 \end{array}\right] \quad \tfrac{1}{31} R_3$$

Toggle slides back and forth to compare before and after matrix changes

Example

Use the Gauss-Jordan elimination method to solve the system of equations

$$\begin{aligned} 3x - 2y + 8z &= 9 \\ -2x + 2y + z &= 3 \\ x + 2y - 3z &= 8 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 0 & 9 & 12 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_1 - 9R_3 \\ \\ R_2 + 6R_3 \end{array}$$

Toggle slides back and forth to compare before and after matrix changes

Example

Use the Gauss-Jordan elimination method to solve the system of equations

$$\begin{aligned} 3x - 2y + 8z &= 9 \\ -2x + 2y + z &= 3 \\ x + 2y - 3z &= 8 \end{aligned}$$

Solution

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_1 - 9R_3 \\ R_2 + 6R_3 \end{array}$$

Toggle slides back and forth to compare before and after matrix changes

Example

Use the Gauss-Jordan elimination method to solve the system of equations

$$\begin{aligned} 3x - 2y + 8z &= 9 \\ -2x + 2y + z &= 3 \\ x + 2y - 3z &= 8 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

The solution to the system is thus $x = 3$, $y = 4$, and $z = 1$.

5.3

SYSTEMS OF LINEAR EQUATIONS:

UNDERDETERMINED AND OVERDETERMINED SYSTEMS

$$\begin{aligned} x + 2y - 3z &= -2 \\ 3x - y - 2z &= 1 \\ 2x + 3y - 5z &= -3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 3 & -1 & -2 & 1 \\ 2 & 3 & -5 & -3 \end{array}\right]$$

$$\begin{aligned} x - z &= 0 \\ y - z &= -1 \end{aligned} \qquad \begin{aligned} x &= z \\ y &= z - 1 \end{aligned}$$

A System of Equations with an Infinite Number of Solutions

Solve the system of equations given by

$$\begin{aligned} x + 2y - 3z &= -2 \\ 3x - y - 2z &= 1 \\ 2x + 3y - 5z &= -3 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 3 & -1 & -2 & 1 \\ 2 & 3 & -5 & -3 \end{array}\right] \begin{array}{l} R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

Toggle slides back and forth to compare before and after matrix changes

A System of Equations with an Infinite Number of Solutions

Solve the system of equations given by

$$\begin{aligned} x+2y-3z&=-2\\ 3x-\ \ y-2z&=\ \ 1\\ 2x+3y-5z&=-3 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & -7 & 7 & 7 \\ 0 & -1 & 1 & 1 \end{array}\right] \begin{array}{l} R_2-3R_1 \\ -\frac{1}{7}R_2 \\ R_3-2R_1 \end{array}$$

Toggle slides back and forth to compare before and after matrix changes

A System of Equations with an Infinite Number of Solutions

Solve the system of equations given by

$$\begin{aligned} x + 2y - 3z &= -2 \\ 3x - y - 2z &= 1 \\ 2x + 3y - 5z &= -3 \end{aligned}$$

<u>Solution</u>

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 1 & 1 \end{array}\right] \begin{array}{l} R_1 - 2R_2 \\ -\frac{1}{7}R_2 \\ R_3 + R_2 \end{array}$$

Toggle slides back and forth to compare before and after matrix changes

A System of Equations with an Infinite Number of Solutions

Solve the system of equations given by

$$x + 2y - 3z = -2$$
$$3x - y - 2z = 1$$
$$2x + 3y - 5z = -3$$

<u>Solution</u>

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_1 - 2R_2 \\ R_3 + R_2 \end{array}$$

Toggle slides back and forth to compare before and after matrix changes

A System of Equations with an Infinite Number of Solutions

Solve the system of equations given by

$$\begin{aligned} x + 2y - 3z &= -2 \\ 3x - y - 2z &= 1 \\ 2x + 3y - 5z &= -3 \end{aligned}$$

<u>Solution</u>

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Observe that row three reads 0 = 0, which is true but of no use to us.

A System of Equations with an Infinite Number of Solutions

Solve the system of equations given by

$$\begin{aligned} x + 2y - 3z &= -2 \\ 3x - y - 2z &= 1 \\ 2x + 3y - 5z &= -3 \end{aligned}$$

<u>Solution</u>

This last augmented matrix is in row-reduced form.

Interpreting it as a system of equations gives a system of two equations in three variables *x*, *y*, and *z*:

$$\begin{aligned} x - z &= 0 \\ y - z &= -1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

A System of Equations with an Infinite Number of Solutions

Solve the system of equations given by

$$
\begin{aligned}
x + 2y - 3z &= -2 \\
3x - y - 2z &= 1 \\
2x + 3y - 5z &= -3
\end{aligned}
$$

<u>Solution</u>

Let's single out a single variable –say, *z*– and solve for *x* and *y* in terms of it.

If we assign a particular value of *z* –say, *z* = 0– we obtain *x* = 0 and *y* = –1, giving the solution (0, –1, 0).

$$
\begin{aligned}
x - z &= 0 \qquad & x &= z = (0) = 0 \\
y - z &= -1 \qquad & y &= z - 1 = (0) - 1 = -1
\end{aligned}
$$

A System of Equations with an Infinite Number of Solutions

Solve the system of equations given by

$$x + 2y - 3z = -2$$
$$3x - y - 2z = 1$$
$$2x + 3y - 5z = -3$$

<u>Solution</u>

Let's single out a single variable – say, *z* – and solve for *x* and *y* in terms of it.

If we instead assign *z* = 1, we obtain the solution (1, 0, 1).

$$x - z = 0 \qquad x = z = (1) = 1$$
$$y - z = -1 \qquad y = z - 1 = (1) - 1 = 0$$

A System of Equations with an Infinite Number of Solutions

Solve the system of equations given by

$$\begin{aligned} x + 2y - 3z &= -2 \\ 3x - y - 2z &= 1 \\ 2x + 3y - 5z &= -3 \end{aligned}$$

<u>Solution</u>

Let's single out a single variable –say, *z*– and solve for *x* and *y* in terms of it.

In general, we set *z* = *t*, where *t* represents any real number (called the parameter) to obtain the solution (*t*, *t* – 1, *t*).

$$x - z = 0 \qquad x = z = (t) = t$$

$$y - z = -1 \qquad y = z - 1 = (t) - 1 = t - 1$$

A System of Equations That Has No Solution

Solve the system of equations given by

$$\begin{aligned} x + y + z &= 1 \\ 3x - y - z &= 4 \\ x + 5y + 5z &= -1 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & -1 & -1 & 4 \\ 1 & 5 & 5 & -1 \end{array}\right] \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 - R_1 \end{array}$$

Toggle slides back and forth to compare before and after matrix changes

A System of Equations That Has No Solution

Solve the system of equations given by

$$
\begin{aligned}
x + y + z &= 1 \\
3x - y - z &= 4 \\
x + 5y + 5z &= -1
\end{aligned}
$$

Solution

$$
\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -4 & -4 & 1 \\ 0 & 4 & 4 & -2 \end{array}\right]
\quad
\begin{array}{l} R_2 - 3R_1 \\ R_3 + R_2 \\ R_3 - R_1 \end{array}
$$

Toggle slides back and forth to compare before and after matrix changes

A System of Equations That Has No Solution

Solve the system of equations given by

$$\begin{aligned} x + \ y + \ z &= \ 1 \\ 3x - \ y - \ z &= \ 4 \\ x + 5y + 5z &= -1 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -4 & -4 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right] \quad R_3 + R_2$$

Toggle slides back and forth to compare before and after matrix changes

A System of Equations That Has No Solution

Solve the system of equations given by

$$\begin{aligned} x + y + z &= 1 \\ 3x - y - z &= 4 \\ x + 5y + 5z &= -1 \end{aligned}$$

<u>Solution</u>

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -4 & -4 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

Observe that row three reads $0x + 0y + 0z = -1$ or $0 = -1$!

We therefore conclude the system is inconsistent and has no solution.

- If there is a row in the augmented matrix containing all zeros to the left of the vertical line and a nonzero entry to the right of the line, then the system of equations has no solution.