# On first countable and minimal topological spaces

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## ABSTRACT

**In this paper, we study the concept of minimal topological spaces and its relation with first countable space, we prove that if X is first countable completely regular space, then the following are equivalent.**

- **X is first countable and minimal completely regular space.**
- **X is first countable and completely regular – closed space.**

**A first countable and minimal Urysohn is semi regular and Let $\{x(n): n \in M\}$ be a collection of a topological spaces and $X = \Pi\, x(n)$, then X is first countable and Hausdorff – closed if and only if each *x(n)* is first countable and Hausdorff – closed.**

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## Introduction

If p is property of topologies a space(x, τ) is called minimal p or p-minimal if τ has property p but no topology on x which is strictly weaker than τ has p . (x, τ) is p-closed if τ has property p and (x, τ) is closed subspace of every p-space in which it can be embedded .

Minimal p and p-closed spaces have been investigated for the cases p=Hausdorff, Regular, Urysohn, completely Hausdorff. A well known result is that for any of these properties a compact p-space is minimal p. the concept of minimal topologies was first introduced in 1939 by A.S. Parhomenko when he showed that compact Hausdorff are minimal Hausdorff spaces in 1947 A. Ramanathan gave all characterization of minimal Hausdorff spaces, In his book "Topological structures" W.j. Thron proved that a first countable, Hausdorff countable compact space is minimal first countable Hausdorff space.

**Definition 1[6]**

A filter base on a topological space is called open (closed) provided that the sets belonging to it are open (closed)

**Definition 2[6]**

A regular filter base is an open filter base which is equivalent with a closed filter base.

**Definition 3[6]**

**A topological space X is feebly compact if every countable open filter base has an a adherent point .**

**Definition 4[6]**

**Let .(X, $\tau$) be topological space and A Í X the intersection of all closed supper sets of A is called the closure of A which is denoted by Cl(A)**

**Definition 5[6]**

**Let .(X, $\tau$) be topological space and A Í X, Appoint x$\in X$is said to be an interior point of A if and only if A is a neighborhood of x.**

**The set of all interior points of A are called the interior of A which is denoted by Int (A).**

**Definition 6[7]**

**Given a space (X, $\tau$) ,{Int $\tau$ cl $\tau$ T/T$\in \tau$ }is a base for a topology $\tau_S$ on X .(X, $\tau$) is called semi regular if $\tau_S= \tau$**

**Definition 7[7]**

**A topological space (X, $\tau$) is said to be regular if and only if every closed set F and every P$\notin$F there are disjoint open sets G and H in X such that F Ì G, P Î H.**

**Definition 8[6]**

**Let .(X, $\tau$) be a topological space, we say that (x, $\tau$) is completely regular space if and only if for every closed set F $\subseteq$ X, and for all x$\in X$ such that x$\notin$ F there is continuous function f: X$\to$[0,1] such that f(x)=0 f[F]=1.**

**Definition 9[6]**

**Let .(X, $\tau$) be a topological space, we say that Urysohn space if and only if for all x,y $\in$ X such that x $\neq$ y there are open sets G,H in X such that F $\subseteq$ G, X $\in$ H.**

**Definition 10[6]**

**Let .(X, $\tau$) be a topological space we say that (X, $\tau$) is Hausdorff space if and only if for all x,y $\in$ X such that x $\neq$ y there are disjoint open sets G,H such that x $\in$ G, y $\in$ H.**

**Definition 11[6]**

**Let .(X, $\tau$) be a topological space we say that (X, $\tau$) is first countable space if and only if there is accountable local base at each of its points.**

**Theorem 1 [5]**

**Let $X$ be a first countable Hausdorff space . the following are equivalent**

**(1)X is first countable and minimal Hausdorff**

**(2)X is semi regular and feebly compact**

**(3)Every countable open filter base on X which has a unique adherent point is convergent**

**Theorem 2 [5 ]**

**Let (x, $\tau$) be a first countable Hausdorff space the following are equivalent**

**(1) (x, $\tau$) is first countable and Hausdorff-closed**

**(2) (x, $\tau$) is feebly compact**

**(3) (x, $\tau_S$) is first countable and minimal Hausdorff**

**Theorem 3**

**Let X be first countable regular Hausdorff space .the following are equivalent**

**(1) X is first countable and minimal regular**

**(2) Every countable regular filter base on X which has unique adherent point is convergent**

**(3)Every countable regular filter base on x has an adherent point.**

**(4) Every countable regular filter base is fixed.**

**(5) X is first countable and regular closed**

**(6) X is feebly compact .**

**(7) X is first countable and minimal Hausdorff.**

**Proof:**

**proofs that (1)and (2) are equivalent ,that (3) and (5) are equivalent and that (2) implies (3) in [5] and [4], (3) and (4)since for any regular filter base $\beta$, $\cap \beta$ ={ $\cap$ B/B$\in \beta$},in[8] proved that regular space is feebly compact if and only if (4)holds.**

if (4) holds then X satisfies (2) (Theorem1),Hence X is first countable and minimal Hausdoff ,also its clear that (6)implies to (1)by [1]

**Definition 12[7]**

An open filter base $\beta$ on a space X is called completely regular if for each B $\in \beta$ there exist a set C $\in \beta$ and continues function f:X→[0,1] such that f (C) =0 and f(X-B) =1

**Theorem 4**

Let (x, $\tau$) be a completely regular space ,and let $\beta$ be a countable regular filter base on (x, $\tau$) . fix a point p$\in$X and let $\tau^{\backslash}$ be the topology on X which has a base $\tau$ / (X-{P}) $\cup${T$\cup$C / p$\in$T ,T$\in \tau$ and C$\in \beta$} then $\tau^{\backslash}$ is a completely regular topology on X which is strictly weaker than ( X, $\tau^{\backslash}$)is first countable if and only if (X, $\tau$) is first countable .

Proof : we'll prove that $\tau^{\backslash}$ is a completely regular topology on X which is strictly weaker than $\tau$. Since $\beta$ is regular filter base on (X, $\tau$) is, $\beta$ has no adherent points thus (X, $\tau^{\backslash}$) is Hausdorff. also $\tau^{\backslash}$ is weaker than $\tau$ since $\beta$ is an open filter base on (X, $\tau$) . if T is an open neighborhood of P in (X, $\tau$) which is disjoint from a set C$\in \beta$ then T contains no neighborhood P in (X, $\tau^{\backslash}$) therefore $\tau^{\backslash}$ is strictly weaker that $\tau$ .

**To prove that (X, $\tau^{\backslash}$) is completely regular , consider a point x $\in$X and an open neighborhood N of x in (X, $\tau^{\backslash}$)**

**Case1 : let x$\neq$p , since x is not an adherent point of $\beta$ in (X, $\tau$) there is an neighborhood M of x in X – { P} therefore C$\in \beta$ , M$\cap$ C =$\emptyset$ , let H= M$\cap$N, Since (X, $\tau$) is completely regular , There is a continues function**

**f: (X, $\tau$) $\rightarrow$ [0,1] which vanishes at x and f (X-H ) =0 hence f is continuous on (x, $\tau^{\backslash}$) , f(x) =0 and f (X-N) =1**

**case 2 : let x=p . we choose T $\in$ $\tau$ and C $\in$ $\tau$ and C$\in \beta$ that x$\in$T and N$\supset$T$\cup$C then there exist continues mapping f, of (X,$\tau$) into [0,1] and a let D $\in \beta$ such that f (x) =0 , f(X-T) =1 ,g(D) =0 and g(X-C) =1 , let y=f$\wedge$ g Then h: (X, $\tau^{\backslash}$) $\rightarrow$[0,1] Such that h(x)=o,h (X-N) =1**

**Theorem 5**

**Let X be a first countable completely regular space the following are equivalent (1) X is first countable and minimal completely regular (2) X is first countable and closed completely regular (3) X is pseudo compact (4) X is feebly compact (5) Every completely regular filter base on X fixed .**

**Proofs :**

**To prove (1) implies (2) , let Y be a first countable completely regular space . Such that X$\subseteq$ Y and support that q$\in \bar{X}$ . let D be a countable fundamental system of open neighborhoods of q then**

**$\beta = D/X$ is countable completely regular filter base on X , so by theorem (4) there is a point x $\epsilon$X and X $=\overline{X}$ .**

**Also (2) , (3) , (5) are equivalent in [9] , it well known that (3) and (4) are equivalent [7] , According to theorem [3]. (4) implies that X is first countable and minimal Hausdorff thus (4) implies (1).**

**Theorem 6**

**A first countable and minimal urysohn space (X, $\tau$) is semi regular .**

**Proof :-**

**Since (X, $\tau$) is a urysohn space then (X, $\tau_s$) is semi regular space [10] , since $\tau_s \subset \tau$ and (X, $\tau_s$) is first countable if (X, $\tau$) is first countable , we must have $\tau = \tau_s$ if (X, $\tau$) is first countable and minimal urysohn , Hence (X, $\tau$) is semi regular .**

**Theorem 7 :**

**let { X (n) : n $\in$ M } be a collection of topological spaces and X = $\pi$ X (n) . Then x is first countable and Hausdorff - close if and only each X (n) is first countable and Hausdorff -closed**

**Proof :**

**Since X is first countable if and only if each X (n) is first countable its follows from theorem 4.2 and 4.4 in [8] that X is feebly compact if and only if each X (n) is feebly compact .**

**Since X is first countable and feebly compact then X is first countable and Hausdorff closed by theorem [2] , hence the proof is done .**

**As far as the author knows , the following question is open : what are necessary and sufficient conditions that the product of a countable collection of spaces be first countable and minimal urysohn , (urysohn –closed ) ? the next few results give some partial answers .**

**<u>Theorem 8</u> :**

**if { X(n) n∈ M } is a collection of spaces such that X= ΠX (n) is first countable and minimal urysohn if and only if each X (n) is a first countable and minimal urysohn space .**

**<u>Proof</u> :**

**This is an immediate consequence of the fact that a collection { X (a) \a∈A } of space has aurysohn product if and only if each X(a) is a urysohn space .**

## <u>Lemma1[5]</u>

**Let X be first countable urysohn space the following are equivalent**

1. **X is first countable and urysohn –closed .**

2. **Whenever $\propto$ and $\beta$ are arbitrary open filter bases onX, $\psi$ is accountable closed filter base on X , $\propto$ is weaker than $\psi$ and $\psi$ is weaker than $\beta$ , then $\propto$ has non empty adherence point .**

**Theorem 9**

**If X and Y are first countable and urysohn – closed spaces and X is Hausdroff- closed then X×Y is a first countable and urysohn – closed .**

**Proof :**

**we first note that for every open sub set A of X×Y , $pr_2$ ($\bar{A}$) is closed [8]**

**$let$α and $\beta$ be an open filter bases on X×Y such that α is weaker than countable filter base $\psi$ = {$\bar{L}$ : L $\epsilon$/ $\beta$ } than $pr_2$ (α) , $pr_2$ ($\psi$) , $pr_2$ ($\beta$) satisfy hypothesis (2) of lemma 1, So $pr_2$ (α) has an adherent point y ,let $\beta$ be a countable fundamental system of open neighborhoods of y and let D ={ J⋂(X×C) /J∈ α and C ∈ $\beta$} then pr (D) is an open filter base and hence has an adherente point x , The point ( x ,y) is an adherent point α .**

## Conclusion

**I) if X is first countable completely regular space, then the following are equivalent.**

**X is first countable and minimal completely regular space.**

**X is first countable and completely regular – closed space.**

**II) A first countable, minimal Urysohn space is a semi – regular.**

**III) if {X(n)} is a sequence of topological spaces then Π X(n) if first countable and minimal p if and only if each X(n) is first countable and minimal P.**

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# حول الفضاء المعدود الأولي والفضاءات التبولوجية الأصغرية

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## المستخلص

**في هذا البحث درسنا مفهوم الفضاءات التبولوجية الأصغرية وعلاقتها بالفضاء المعدود الأولي وبرهنا إذا كانت X فضاء معدود أولي ومنتظم كامل فإن العبارتين التاليتين متكافئتين:**

- X فضاء معدود أولي ومنتظم كامل أصغري.
- X فضاء معدود أولي ومنتظم كامل – مغلق.

**وبرهنا أي فضاء معدود أولي ويوريزون أصغري فإنه يكون فضاء شبه منتظم.**
**ولتكن $\{x(n): n \in M\}$ عائلة من الفضـاءات التبولوجيـة و,$X = \Pi\, x(n)$ فإن X فضاء معدود أولي وهاوسدورف – مغلق إذا وفقط إذا كان X(n) فضاء معدود أولي وهاوسدورف – مغلق.**

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