On The Solution of Discrete-Time Linear Operator Equations

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ABSTRACT

In this paper, we introduce and discuss the existence and uniqueness of the solution of discrete-time Sylvester and Lyapunov operator equations. And, we study the nature of these operator equations for special types of operators.

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1-Introduction

An operator equation of the form

$L(X) = W$,

is said to be an operator equation, where L and W are known operators define on a Hilbert space H, and X is the unknown operator that must be determined.

In the above operator equation, if the operator L is linear then this equation is called linear operator equation, otherwise, it is a non- linear operator equation.

Linear operator equations are very important in control theory and many other branches of engineering,[2].

Many authors studied the operator equation foe example Golden J. in 1978 studied the existence and uniqueness of the solution for the linear operator equation of the form

$AX + XB = Q$, where A, B and Q are known operators define on a Hilbert space H, and X is the unknown operator that must be determine, [6].

Bahatia and Rosenthal. in 1997 illustrate, the importance of the study of the previous linear operator equation,[2]. Also, in 2001 Bahatia studied a special type of linear operator equations of the form

$A^*X + XA + tA^*XA^{1/2} = W$, where A and W are known operators defined on H, t is any scalar and X is the unknown operator,[1].

In 2005 Emad A.K. studied a special type of linear operator equations (Lyapunov equation) of the form $A^*X + XA = W$,where A and $W$are known operators defined on H and A^* is the adjoint of A, X is the unknown operator,[5].

2. Some Types of Linear Operator Equations:

In this section some types of linear operator equations are introduced:

I. Continuous and discrete-time Sylvester operator equations:

$$AX \pm XB = \alpha C , \qquad \dots(1)$$

$$AXB \pm X = \alpha C . \qquad \dots(2)$$

II. Continuous and discrete-time Lyapunov operator equations:

$$A^*X - XA = \alpha C \qquad \dots(3)$$

$$A^*XA - X = \alpha C \quad . \quad \dots(4)$$

where A, B and C are given operators defined on a Hilbert space H, X is an operator that must be determined, α is any scalar, and A^* is adjoint of the operator A, [1].

In general these linear operator equations may have one solution, infinite set of solutions or no solution.

3. The Existences and Uniqueness of The Solution of The Discrete-Time Operator Equations:

Existence and uniqueness of the solution of eq.'s(2) and (4), when B is an invertible operator in eq.(2) and A is an invertible operator in eq.(4) are studied,[4].

The discrete-time Sylvester equation can be transform to continuous-time Sylvester equation as follows:

Multiply eq. (2) from the right by B^{-1},then eq.(2) becomes:

$$AX \pm XB^{-1} = \alpha CB^{-1}$$

Let $CB^{-1} = W$, the above equation becomes:

$$AX \pm XB^{-1} = \alpha W \qquad \ldots(5)$$

Also, the discrete-time Lyapunov operator equation can be transform to continuous-time operator equation as follows:
Multiply eq.(4) from the right by A^{-1}, then eq.(4) becomes:

$$A^{*}X - XA^{-1} = \alpha W \quad , \qquad \ldots(6)$$

Recall that, the spectrum of the operator $A \equiv \sigma(A) = \{\lambda \in \mathbb{C}; (A - \lambda I)$ is not invertible} and $B(H)$ is the Banach space of all bounded linear operators defined on the Hilbert space,[3].

Corollary (3.2),[4]:
If A and B are operators in $B(H)$, and B^{-1} exist, such that $\sigma(A) \cap \sigma(B^{-1}) = \emptyset$, then the operator equation $AX - XB^{-1} = \alpha W$, has a unique solution X, for every operator W.

Corollary (3.3)[4]:
If A and B are operators in $B(H)$, and B^{-1} exist, such that $\sigma(A) \cap \sigma(-B^{-1}) = \emptyset$, then the operator equation $AX + XB^{-1} = \alpha W$, has a unique solution X, for every operator W.

Corollary (3.4),[4]:
If A an operator in $B(H)$, A^{-1}exist such that $\sigma(A^{*}) \cap \sigma(A^{-1}) = \emptyset$,then eq.(6) has a unique solution X, for every operator W.

Proposition (3.5):
consider eq.(6), if $\sigma(A^{*}) \cap \sigma(A^{-1}) = \emptyset$, then the operator $\begin{bmatrix} A^{*} & -\alpha W \\ 0 & A^{-1} \end{bmatrix}$ is defined on $H_1 \oplus H_2$ is similar to the operator $\begin{bmatrix} A^{*} & 0 \\ 0 & A^{-1} \end{bmatrix}$.

Proof:

Since $\sigma(A^*) \cap \sigma(A^{-1}) = \emptyset$, then by Sylvester-Rosenblum theorem, eq.(6), has a unique solution .Also

$$\begin{bmatrix} I & X \\ o & I \end{bmatrix}\begin{bmatrix} A^* & 0 \\ 0 & A^{-1} \end{bmatrix} = \begin{bmatrix} A^* & -\alpha W \\ 0 & A^{-1} \end{bmatrix}\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$$

But

$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$ is invertible, so $\begin{bmatrix} A^* & 0 \\ 0 & A^{-1} \end{bmatrix}$ is similar to $\begin{bmatrix} A^* & -\alpha W \\ 0 & A^{-1} \end{bmatrix}$.

the converse of the above proposition is not true in general.

4. The Nature of The Solution for the Discrete-Time Lyapunov Operator Equation.

In this section, we study the nature of the solution for special types of the linear operator equation, namely the discrete-time Lyapunov equation.

Remarks (4.1):

1. If A, A^{-1} and W are self-adjoint operators, then eq.(6), may or may not have a solution Moreover, if it has a solution then it may be non self-adjoint.
2. Consider eq.(6), if W is self-adjoint operator, then it is not necessarily that $X = X^*$.
3. If A, A^{-1} and W are skew-adjoint operators, then eq.(6) has no

Solution.

These remarks can be easily be observed in matrices.

Remarks (4.2):

1. If A and W are normal operators, then the solution X is not necessarily normal operator.
2. If W is normal operator and A is any operator, then it is not necessarily that the solution X is normal operator.

Putnam- Fugled Theorem (4.3):

Assume that $M, N, T \in B(H)$, where M and N are normal.
If $MT = TN$ then $M^*T = TN^*$.
Proof: see [7].
Recall that, an operator M is said to be dominant if$\|(T-Z)^* x\| \leq M_z\|(T-Zx)\|$, for all Z$\in \sigma(T)$ and x $\in H$, [3].
Also, an operator M is called M- hyponormal operator if $\|(T-Z)^*x\| \leq M\|(T-Z)x\|$, for all $Z \in \mathbb{C}$ and $x \in H$,[3].

Theorem (4.4),[7]:
Let M be dominant operator and N^* is an M-hyponormal operator.Assume that $MT = TN$ for some $T \in B(H)$ then $M^*T = TN^*$.

Theorem (4.5),[7]:
Let A and B be tow operators that satisfy Putnam-Fugled condition. The operator equation $AX - XB = C$ has a solution X if and only if $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ are similar operator on $H_1 \oplus H_2$.
According to above theorems, we have the following corollaries:

Corollary (4.6):
If A is normal operator and A^{-1} exists then operator equation $A^*X + XA^{-1} = \alpha W$ has a solution if and only if $\begin{bmatrix} A^* & 0 \\ 0 & -A^{-1} \end{bmatrix}$ is similar to $\begin{bmatrix} A^* & -\alpha W \\ 0 & -A^{-1} \end{bmatrix}$.

Corollary (4.7):
If A and B are normal operators and B^{-1} exists then the operator equation $AX - XB^{-1} = \alpha W$ has a solution if and only if $\begin{bmatrix} A & 0 \\ 0 & -B^{-1} \end{bmatrix}$ is similar to $\begin{bmatrix} A & -\alpha W \\ 0 & -B^{-1} \end{bmatrix}$.
The following corollaries follows directly from theorem (4.4).

Corollary (4.8):

If A is a dominant or M-hyponormal operator and A^{-1} exists, then the operator equation $A^*X + XA^{-1} = \alpha W$ has a solution if and only if $\begin{bmatrix} A^* & 0 \\ 0 & -A^{-1} \end{bmatrix}$ and $\begin{bmatrix} A^* & -\alpha W \\ 0 & -A^{-1} \end{bmatrix}$ are similar operator on $H_1 \oplus H_2$.

Corollary (4.9):

If A and B are dominant or M-hyponormal operators and B^{-1} exists. Then the operator equation $AX - XB^{-1} = \alpha W$ has a solution if $\begin{bmatrix} A & 0 \\ 0 & -B^{-1} \end{bmatrix}$ and $\begin{bmatrix} A & -\alpha W \\ 0 & -B^{-1} \end{bmatrix}$ are similar operators on $H_1 \oplus H_2$.

Proposition (4.10):

Consider eq.(6), if A is an orthogonal operator, A^{-1} exists and W is also an orthogonal operator, and the solution X of eq.(6) is unique then this solution is orthogonal.

Proof:

Consider the operator equation:

$A^*X + XA^{-1} = W$,

then $(A^*X + XA^{-1})^* = W^*$,

Since W is an orthogonal operator $(W^* = W^{-1})$ implies that $W = (W^{-1})^*$.

$X^*A + (A^{-1})^*X^* = W^*$,

$[X^*A + (A^{-1})^*X^* = W^*]^{-1}$,

Since A is an orthogonal operator $(A^* = A^{-1})$

$A^{-1}(X^*)^{-1} + (X^*)^{-1}A^* = (W^*)^{-1}$,

$A^*(X^*)^{-1} + (X^*)^{-1}A^{-1} = W$,

then $(X^*)^{-1} = X$ **, So** $X^* = X^{-1}$.

Therefore, X is an orthogonal operator.

Proposition (4.11):

Consider eq.(6), if A is unitary operator and W is orthogonal operator and the solution of eq.(6) is unique then this solution is orthogonal operator.

Proof:

Consider the following linear operator equation:

$A^*X + XA^{-1} = W$

$(A^*X + XA^{-1})^* = W^*$

$X^*A + (A^{-1})^*X^* = W^*$

$(X^*A + (A^{-1})^*X^*)^{-1} = (W^*)^{-1},$

$A^{-1}(X^*)^{-1} + (X^*)^{-1}[(A^{-1})^*]^{-1} = (W^*)^{-1},$

Since A is unitary operator (every orthogonal operator is a unitary) then $A^* = A^{-1}$.

So $A^*(X^*)^{-1} + (X^*)^{-1}A^{-1} = (W^*)^{-1},$

Since eq. (6) has a unique solution, then $X = (X^*)^{-1} = (X^{-1})^*$.therefore $X^* = X^{-1}$,

implies that X is an orthogonal operator.

Remark (4.12):

If A is a skew-adjoint and W is self –adjoint and the operator equation $A^*X + XA^{-1} = W$ has only one solution, then this solution is not necessarily a skew- adjoint or self- adjoint.

Remark (4.13):

If A and W are skew-adjoint operators, the operator equation $A^*X + XA^{-1} = W$ has only one solution then this solution is not necessarily self-adjoint.

These remarks can be easily seen in matrices.

5-Conclusion:

In this paper conclude :

-The nature of the solution depend on the known operators .

-The existences and uniqueness of the operator equation depend on (for special cases) the type known operator (dominant operator, M-hyponormal operator, normal operator,...).

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حول حلول المعادلات الخطية المتقطعة المؤثرة

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المستخلص

في هذا البحث قدمنا وناقشنا وجود ووحدانية الحل لمعادلات ليبانوف وسلفستر المتقطعة المؤثرة. كما تم دراسة طبيعة الحل لتلك المعادلات المؤثرة لانواع خاصة من المؤثرات .

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