On The Solutions Of Quasi – Lyapunov Operator Equations

Asst.Prof.Dr.. Emad A. Kuffi *

Abstract

In this paper, some theorems are modified to ensure the existence and uniqueness of the solution for the Quasi – Lyapunov operator equation.

Also, the range of the Quasi – Lyapunov equation is studied. As well as, the nature of the solution for the Quasi – Lyapunov operator equation is studied for special types of operators.

Keywords: Lyapunov equations, Sylvester equation, Linear operator equations, Sylvester – Rosenblum Theorem.

* Al-Mansour University College

(1) Introduction

In this introduction, we give some types of linear operator equations:
(1) The linear operator equation of the form

$$A^*X + XA = W, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Where A and W are given operators defined on Hilbert space H, and X is the unknown operator that must be determined. This linear operator equation is called the Lyapunov operator equation, or the continuous – time Lyapunov equation, [3] and [5].

The author in reference [3] studied the necessary and sufficient conditions for the solvability of this linear operator equation.

(2) A special case of the continuous – time Lyapunov operator equation

$$AX + XA = W, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Where A and W are known operators defined on a Hilbert space H, and X is the unknown operator that must be determined, [3] and [4].

(3) The linear operator equation of the form

$$AX + X^*A = W, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Where A and W are given operators defined on a Hilbert space H, and X is the unknown operator that must be determined, X^* is the adjoint of X. These linear operator equations (2) and (3) are called quasi Lyapunov operator equations or quasi – continuous – time Lyapunov linear operator equations.

(2)_The Quasi – Continuous – Time Lyaponov Operator Equations:

The continuous – time Lyapunov equations, are much studied because of it's importance in differential equations and control theory, [6]. Therefore we devote the studying of the quasi continuous – time Lyapunov operator equations.

Now, does eq. (2) and eq. (3) have a solution?

If yes, is it unique?

To answer this equation, recall the Sylvester – Rosenblum theorem, [5].

<u>Sylvester – Rosenblum Theorem (2.1):</u>

If A and B are operators in B(H) such that $\sigma(A)\cap\sigma(B) = \Phi$, then AX-XB = Y has a unique solution X fore every operator Y.

According to the Sylvester – Rosenblum theorem, we have the following corollary:

<u>Corollary (2.1):</u>

If A is an operator such that $\sigma(A)\cap\sigma(-A) = \Phi$, then eq. (2) has a unique X for every operator W.

<u>Proposition (2.1):</u>

Consider eq. (2), if $\sigma(A)\cap\sigma(-A) = \Phi$, then

The operator $\begin{bmatrix} A & -W \\ 0 & -A \end{bmatrix}$ is defined on $H_1 \oplus H_2$ is similar to $\begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix}$.

<u>Proof:</u>

Since $\sigma(A)\cap\sigma(-A) = \Phi$. Then by Sylvester – Rosenblum theorem eq. (2) has a unique solution X, also:

$$\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} = \begin{bmatrix} A & -W \\ 0 & -A \end{bmatrix} \begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}.$$

But $\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}$ is invertible so $\begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix}$ is similar to $\begin{bmatrix} A & -W \\ 0 & -A \end{bmatrix}$.

The converse of the above proposition is not true in general as we see the following example.

Example:

Let H = ℓ_2 (C), that is,

ℓ_2 (C) = $\left\{ X=(x_1,x_2,...) : \sum_{i=1}^{\infty} |x_i|^2 < \infty,\ x_i \in C \right\}$

Define A: H⟶H by A $(x_{1,}\ x_2,...)=(x_1,0,0,...)$. Consider eq. (2), where W$(x_{1,}\ x_2,...)=(0,x_1,0,...)$. Then X = U is A solution of this equation since $AX+XA=(x_{1,}\ x_2,...)=(AU+UA)(x_{1,}\ x_2,...)$

$$A(0,x_1,x_2,...)+U(x_1,0,0,...)=(0,0,0,...)+(0,x_1,0,...)=(0,x_1,0,0)=WX$$

.

On the other hand, U is solution of eq. (2) and

$$\begin{bmatrix} 1 & U \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} = \begin{bmatrix} A & -W \\ 0 & -A \end{bmatrix} \begin{bmatrix} 1 & U \\ 0 & 1 \end{bmatrix}.$$

Therefore, $\begin{bmatrix} A & -W \\ 0 & -A \end{bmatrix}$ is similar to $\begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix}$.

Moreover 0 is an eigenvalue of A and X= $(0, x_2, ...)$ is the associated eigenvector.

Therefore, 0 ∈ σ(A)∩σ(– A) and hence σ(A)∩σ(– A) ≠ Φ

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(3) The Nature of the solution of Quasi – Continuous – Time Lyaponov Operator Equations:

In this paper, we study the nature of the solution of eq. (2) for special types of operators.

Remark (3.1):

If W is self – adjoint operator, and A is any operator, then eq. (2) may or may not have solution. Moreover, if it has a solution then it may be non self – adjoint.

This remark can easily be checked in matrices.

Next if A and W are self – adjoint operators, what condition can one put on A (or W) to ensure the existence of self – adjoint solution for eq. (2).

The following theorem gives one such condition.

Theorem (3.1):

Let A and W be positive self – adjoint operators.
If $0 \notin \sigma(A)$, then the solution X of eq. (2) is self – adjoint.

Proof:

Since $0 \notin \sigma(A)$ then it is easy to see that $\sigma(A) \cap \sigma(-A) = \Phi$ and hence eq. (2) has a unique solution X by Sylvester – Rosenblum theorem.
Moreover,

$$(AX + XA)^* = W^*,$$

$$A^* X^* + X^* A^* = W^*,$$

Since A and W are self – a joint operators, then $AX^* + X^*A = W$.

Therefore, X^* is also a solution of eq. (2). By the uniqueness of the solution on gets $X = X^*$.

Proposition (3.1):

If A and W are self – adjoint operators, and the solution of the eq. (3) exists, then the solution X is a unique.

Proof: Consider

$$AX + X^* A = W ,$$

Since W is self – adjoint operator,

$$(AX + X^* A)^* = W^* ,$$

$$A^* (X^*)^* + X^* A^* = W^* ,$$

Since A is self – adjoint operator,

$AX + X^* A = W$, Since the solution exists, then X is a unique.

The following proposition shows that if operators A and W are skew – adjoint, and the solution of eq. (3) exists then this solution is unique.

Proposition (3.2):

If A and W are skew – adjoint operators, and solution of eq. (3) exists, then the solution X is a unique.

Proof:

Consider eq. (3),

$$AX + X^* A = W ,$$

Since W is a skew adjoint operator, so

$$-(AX + X^* A)^* = -W^* ,$$

$$-\left(A^* (X^*)^* + X^* A^*\right) = -W^* ,$$

$$(-A)X + X^* (-A^*) = -W^* ,$$

Since A and W are skew – adjoint operators, then

$$AX + X^* A = W .$$

Since the solution X exists, then the solution X is a unique.

Remark (3.2):

If A is a self – adjoint operator and W is a skew – adjoint. Then the solution X of eq. (3) is not necessarily exists.

Remark (3.3):

If W is a self – adjoint operator and A is any operator, then the solution X of Eq. (3) is not necessarily self – adjoint operator.

The following example explains this remark.

Example (3.1):

Consider eq. (3), take $W = W^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $A = \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}$,

$$AX + X^*A = W.$$

After simple computations one can gets

$$X = \begin{bmatrix} \alpha & 0 \\ 1 & \frac{1}{2} \end{bmatrix} \neq X^*,$$

Where α is any scalar.

Remark (3.4):

If W is a skew - adjoint and A is any operator, then the solution X of Eq. (3) is not necessarily exists.

The following example explains this remark.

Example (3.2):

Consider eq. (3), take $W=\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ and $A=\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}$,

$$AX+X^{*}A=W.$$

After simple computations one can gets $x_2=1$ and $x_2=0$ which has no solution.

Proposition (3.3):

If A is a compact operator then the eq. (3) is compact.

Proof:

Since A is compact then $X^{*}A$ is also compact.
Since A is compact then XA is also compact.
Since AX and $X^{*}A$ are compact then AX + X^{*}A is compact. Therefore W is compact.

(4) On the Range of ρ_A :

In this section we study and discuss the range of ρ_A ,

ρ(X) = ρ_A (X) = AX+X^{*}A, $X \in B(H)$

Where A is a fixed operator in B(H).
It is clear that the map ρ_A is a linear map. Also the, map ρ_A is bounded, since.

$\|\rho_A\| = \|AX+X^{*}A\| \le \|AX\| + \|X^{*}A\| \le \|A\|\|X\| + \|X^{*}\|\|A\|$

since $\|X^{*}\|$ = $\|X\|$.

Therefore, $\|\rho_A\| \le 2\|A\|\|X\|$,

Let $M = 2\|A\| \ge 0$, so $\|\rho_A\| \le M\|X\|$. Then ρ_A is bounded.

The following steps shows that Range (ρ_A)* ≠ Range (ρ_A),

Range (ρ_A)* = $\left\{\left(AX + X^*A\right)^* \ , \ X \in B(H)\right\}$,

= $\left\{A^*X + X^*A^* \ , \ X \in B(H)\right\}$,

≠ Range (ρ_A) .

Also , α Range (ρ_A) = $\left\{\alpha\left(AX + X^*A\right) \ , \quad X \in B(H)\right\}$

= $\left\{A(\alpha X) + (\alpha X)^* A \ , \ X \in B(H)\right\}$

Let $\alpha X = X_1$

α Range (ρ_A) = $\left\{A^*X_1 + X_1 A \ , \ X_1 \in B(H)\right\}$

= Range (τ_A).

The following remark shows the mapping ρ_A is not – a derivation.

<u>Remark (4.1):</u>

Since ρ_A (XY) = A (XY) + (XY)*A

= A(XY) + Y*X*A .

For all X, Y $\in B(H)$,

And X ρ_A (Y) = [AY + Y*A] ,

= XAY + XY*A .

Also, ρ_A (X)Y = (AX + X*A)Y ,

= AXY + X*AY.

Then one can deduce that:

ρ_A (XY) ≠ X ρ_A (Y) + ρ_A (X)Y .

Now the following remark shows the mapping ρ_A is also not *- a derivation.

<u>Remark (4.2):</u>

Since ρ_A (X + Y) = A(X + Y) + (X + Y) *A ,

$= AX+AY + X^*A + Y^*A$,
$= AX + X^*A + AY + Y^*A$,
$= \rho_A(X) + \rho_A(Y)$.

Now,

$X\rho_A(X) + \rho_A(X)X^* = X[AX + X^*A] + [AX + X^*A]X^*$,
$= XAX + XX^*A + AXX^* + X^*AX^*$,

so $\rho_A(X^2) = \left(AX^2 + (X^*)^2A\right)$,

and $\rho_A(X^2) \neq X\rho_A(X) + \rho_A(X)X^*$.

then ρ_A is not *- a derivation.

(5) On the Range of μ_A (X):

In this section, we study and discuss the range of μ_A , where

$$\mu_A = \mu_A(X) = AX + XA \quad , \quad X \in B(H)$$

Where A is a fix operator in B(H)).

It is clear that the map μ_A is a linear map. Also, the map μ_A is bounded, since $\|\mu_A\| = \|AX + XA\| \le M\|X\|$,

Where $M = 2\|A\| \ge 0$. Then μ_A is bounded.

The following steps shows that Range $(\mu_A)^* \neq$ Range (μ_A).

Range $(\mu_A)^* = \left\{(AX + XA)^* \ , \ X \in B(H)\right\}$,
$= \left\{A^*X^* + X^*A^* \ , \ X \in B(H)\right\}$,
$\neq$ Range (μ_A).

Also, α Range $(\mu_A) = \{\alpha(AX + XA) \ , \ X \in B(H)\}$,
$= \{A(\alpha X) + (\alpha X)A \ , \ X \in B(H)\}$

Let $\alpha X = X_1$

$= \{AX_1 + X_1A \ , \quad X_1 \in B(H)\}$

= Range (μ_A) .

The following remark shows that the mapping μ_A is not a derivation.

<u>Remark (5.1):</u>

Since μ_A (XY) = A(XY) + (XY)A ,

For all X, Y $\in B(H)$,

And X μ_A (Y) = X(AY) + X(YA) ,

= XAY + XYA .

Also, μ_A (X)Y = AXY + XAY ,

Then one can deduce that:

μ_A (XY) ≠ X μ_A (Y) + μ_A (X)Y .

Now, the following remark shows that the mapping μ_A is also not *- a derivation.

<u>Remark (5.2):</u>

Since μ_A (X+Y) = A(X+Y) + (X+Y)A ,

= AX +AY + XA + YA ,

= (AX + XA) + (AY + YA) ,

= μ_A (X) + μ_A (Y) .

Now,

X μ_A (X) + μ_A (X)X^* = X[AX +XA] + [AX + XA]X^*.

= XAX + X^2A + AXX^* + XAX^*.

So, μ_A (X^2) = (AX^2+ X^2A) ,

And μ_A (X^2) ≠ X μ_A (X) + μ_A (X)X^*.

Therefore μ_A is not *- a derivation.

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حول حلول معادلات ليبانوف المؤثرة

أ.م.د. عماد عباس كوفي*

المستخلص

في هذا البحث ،تم تطوير بعض المبرهنات لأثبات وجود وحدانية الحل لمعادلات شبه ليبانوف المؤثرة وطبيعة الحل لمعادلات شبه ليبانوف المؤثرة لأنواع خاصة من المؤثرات.

* كلية المنصور الجامعة